ON THE BERGMAN OPERATORS FOR LINEAR PARTIAL DIFFERENTIAL EQUATIONS

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1. Introduction. The frequent occurrence of the partial differential equation

$$L(U) = U_{z\bar{z}} + aU_z + bU_{\bar{z}} + cU = 0, \tag{1}$$

where a, b, c are functions of $z=x+iy$ and $\bar{z}=x-iy$ and $U_z=\partial U/\partial z$, $U_{\bar{z}}=\partial U/\partial \bar{z}$, $U_{z\bar{z}}=\partial^2 U/\partial z\partial \bar{z}$, in the theory of dynamics has led to a considerable amount of investigation. This investigation has in part been directed along the lines of solving boundary value and characteristic value problems and of finding particular solutions. These problems are of course classical questions in the theory of partial differential equations and have been widely discussed in the literature. Among the recent developments on these problems is the work done by Bergman [1–5],[1] who in his investigations has introduced operators

$$P(f) = \int_{-1}^{+1} E(z, \bar{z}, t) f(z[1 - t^2]/2) dt/(1 - t^2)^{1/2} \tag{2}$$

where f is an arbitrary analytic function of one complex variable. These operators shall be called the Bergman operators. They transform the class $\boldsymbol{A}$ of analytic functions of one complex variable into a certain class of functions $\boldsymbol{C}(E)$, which has the following properties:

(1°) If E is a solution of the equation

$$\begin{aligned} G(E) = {} & (1 - t^2)(E_{z\bar{z}} + aE_{\bar{z}}) - t^{-1}(E_{\bar{z}} + aE) \\ & + 2zt(E_{z\bar{z}} + aE_z + bE_{\bar{z}} + cE) = 0 \end{aligned} \tag{3}$$

which satisfies certain conditions, then every function U of the class $\boldsymbol{C}(E)$ will be a particular solution of $L(U)=0$ and further there always exist solutions E satisfying the mentioned conditions.

(2°) It is possible to determine two functions, say E_1 and E_2, so that $\boldsymbol{C}(E_1)+\boldsymbol{C}^*(E_2)$ represents the totality of solutions of $L(U)=0$ (see [2, 5]). $\boldsymbol{C}^*$ is an analogous class of functions, the f being an analytic function of $\bar{z}$.

The general problem of finding the Bergman operators for a given partial differential equation (1) has been thoroughly discussed by

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[1] The numbers in the brackets refer to the bibliography.

Bergman [1, 2, 3] and by the present author and Ramsay [7]. The singularities of the operators have been treated by Bergman [1] and the present author [6]. In [4] Bergman discussed a method of solution of the classical problems consisting of approximations by expressions $W_n=\sum_{\nu=1}^{n}\alpha_\nu^{(n)}\phi_\nu(x, y)$ where $\phi_\nu(x, y)$ are particular solutions of the considered differential equations. Here the $\alpha_\nu^{(n)}$ are constants which are to be determined by the requirements that the values of W_n on the boundary approximate the given data. (This method is in a certain sense the reverse of the Rayleigh-Ritz method in which the approximating expressions satisfy the boundary conditions but do not satisfy the given equation.) The operators give a simple procedure for the construction of the particular solutions. To further expedite the numerical computation in practical problems it is desirable that the function $E(z, \bar{z}, t)$ have a simple form; for example,

$$E(z, \bar{z}, t) = \exp \sum_{k=1}^{n} C_k(z, \bar{z})t^k. \tag{4}$$

The coefficients a, b, c, and $C_k(z, \bar{z})$ are connected by equation (3). This problem was considered in [7] and in the investigations which lead to that paper the converse problem suggested itself; namely, that of finding partial differential equations for which the Bergman operators have a particular form. In [7] it became apparent that a systematic approach to this converse problem would be to consider in turn expressions of the form $\exp(Nt^n)$, $\exp(Nt^n+Mt^m)$, $\exp(Nt^n+Mt^m+Rt^r)$, and so on, where N, M, and R are functions of z and $\bar{z}$. It is the purpose of this paper to give a general procedure for both problems and to complete the discussion for the case $E=\exp(Nt^n+Mt^m)$.

2. **General procedure.** The equation $L(U)=0$ can always be transformed into the equation

$$L'(V) = V_{z\bar{z}} + BV_z + CV = 0 \tag{5}$$

(see [2, p. 1172]). Thus it suffices to consider this equation. The equation (3) reduces to

$$G(E) = (1 - t^2)(E_{zz}) - E_z/t + 2\bar{z}t(E_{z\bar{z}} + BE_z + CE) = 0. \tag{6}$$

A given form of E will determine a particular solution

$$V = P(f) = \int_{-1}^{+1} E(z, \bar{z}, t)f(z[1 - t^2]/2)dt/(1 - t^2)^{1/2} \tag{7}$$

of the partial differential equation $L'(V)=0$ provided the coefficients B and C satisfy the equation (6) (see [2, p. 1171]). Thus for any given form of E we determine the required derivatives

$$E_1,\ E_{z1},\ E_{\bar{z}1}. \tag{8}$$

Substituting these values into (6) we obtain an equation which is zero for an arbitrary t; thus the coefficients of each power of t must vanish. This yields a system of equations for B, C, M, and N. Upon solving this system we obtain the values for these unknowns and thus determine the partial differential equation (5) for which (7) with the given form of E is a particular solution.

The converse problem then reduces to a discussion of this system. A systematic approach to such a discussion would be to consider in turn the cases in which N is a function of z only (that is, $N_z=0$), M is a function of z only, and so on. If all functions M, N, and so on, are functions of z only, then we have a trivial case. This is easily seen, for consider this case. We have

$$E = \exp \sum_{\nu=1}^{n} C_\nu(z)t^\nu. \tag{9}$$

The required derivatives (8) for the equation (6) are then all zero as each has a derivative with respect to $\bar{z}$. If we further choose $C=0$, the equation (6) is satisfied. This would give the result that the function

$$V = \int_{-1}^{+1} \left[\exp \sum_{\nu=1}^{n} C_\nu(z)t^\nu\right] f(z[1-t^2]/2)\,dt/(1-t^2)^{1/2}$$

is a particular solution of $L'(V)=V_{z\bar{z}}+B(z,\bar{z})V_z=0$ for all values of $B(z,\bar{z})$, $C_\nu(z)$, and ν. But this is evident, as any analytic function $f(z)$ is a solution of this partial differential equation. We shall therefore say that if $N_z=0$, $M_z=0$, and so on, simultaneously, we have a *trivial case.*

3. **Case I.** $E=\exp[N(z,\bar{z})t^n]$. For simplicity of notation let $v=N(z,\bar{z})t^n$. In general we shall let $v=\sum_{k=1}^{n}C_k(z,\bar{z})t_k$ for each case; no ambiguity should arise in its meaning. The required derivatives (8) for the equation (6) are

$$\begin{aligned} E_1 &= e^v[N_1t^n], \\ E_{z1} &= e^v[nNN_zt^{2n-1} + nN_zt^{n-1}], \\ E_{\bar{z}1} &= e^v[N_zN_1t^{2n} + N_{z\bar{z}}t^n]. \end{aligned} \tag{10}$$

The system of equations obtained by substituting (10) into (6) and setting the coefficients of the various powers of t equal to zero is

$$
\begin{aligned}
&t^{n-1}\colon\ (n-1)N_z = 0,\\
(11)\qquad &t^{2n-1}\colon\ nNN_z = 0, \qquad t^{2n+1}\colon\ (2zN_z - nN)N_z = 0,\\
&t^{n+1}\colon\ (2zB - n)N_z + 2zN_{zz} = 0, \qquad t\colon\ 2zC = 0.
\end{aligned}
$$

THEOREM 1. *Exclusive of the trivial case, the only possible values of n in Case* I *for which* (7) *is a particular solution of $L'(V)=0$ are $n=0$, $n=1$, and $n=2$.*

PROOF. To consider any case other than the trivial case N must be a function of $\bar{z}$ as well as z; that is, $N_z \neq 0$. By (11) $N_z \neq 0$ only if $2n-1=X$ where X denotes any exponent of t in (11). Thus

$2n-1=n-1$ or $n=0$; $\quad 2n-1=2n+1$, impossible;

$2n-1=n+1$ or $n=2$; $\quad 2n-1=1$ or $n=1$. q.e.d.

THEOREM 2. *When $n=0$, Case* I *reduces to the trivial case.*

PROOF. If $n=0$, the system (11) reduces to

$$
\begin{aligned}
&t^{-1}\colon\ -N_z = 0,\\
&t\colon\ 2zN_zN_z + 2zN_{zz} + 2BzN_z + 2zC = 0.
\end{aligned}
$$

From the first of these equations $N_z=0$ which is the trivial case.

The cases for $n=1$ and $n=2$ have been discussed in [7] and the solutions for B, C, and N were obtained there.

4. **Case II.** $E=\exp\left[Nt^n+Mt^m\right]=e^v$. If $N=0$, or $M=0$, or $n=m$ this case reduces to Case I; thus we consider only $N\neq 0$, $M\neq 0$, $n\neq m$. In order to avoid further duplication we may, without any loss of generality, order m and n in the following manner

$$
\begin{aligned}
&n < m \text{ if } n \text{ and } m \text{ have different signs;}\\
(12)\qquad &|n| > |m| \text{ in all other cases.}
\end{aligned}
$$

The required derivatives (8) are

$$
\begin{aligned}
E_z &= e^v\left[N_zt^n + M_zt^m\right],\\
E_{zz} &= e^v\left[mM_zt^{m-1} + mMM_zt^{2m-1} + nN_zt^{n-1} + nNN_zt^{2n-1}\right.\\
&\qquad \left. + (mMN_z + nNM_z)t^{n+m-1}\right],\\
E_{z\bar{z}} &= e^v\left[M_{z\bar{z}}t^m + M_zM_{\bar{z}}t^{2m} + N_{z\bar{z}}t^n + N_zN_{\bar{z}}t^{2n}\right.\\
&\qquad \left. + (M_zN_{\bar{z}} + M_{\bar{z}}N_z)t^{n+m}\right].
\end{aligned}
\qquad (13)
$$

The system of equations obtained by substituting (13) into (6) and setting the coefficients of the various powers of t equal to zero is

$$
\begin{aligned}
t^{2n-1}&: \quad nNN_z = 0,\\
t^{n-1}&: \quad (n-1)N_z = 0,\\
t^{2n+1}&: \quad (2zN_z - nN)N_z = 0,\\
t^{n+1}&: \quad (2zB - n)N_z + 2zN_{zz} = 0,\\
t^{2m-1}&: \quad mMM_z = 0,\\
t^{m-1}&: \quad (m-1)M_z = 0,\\
t^{2m+1}&: \quad (2zM_z - mM)M_z = 0,\\
t^{m+1}&: \quad (2zB - m)M_z + 2zM_{zz} = 0,\\
t^{n+m-1}&: \quad mMN_z + nNM_z = 0,\\
t^{n+m+1}&: \quad (2zM_z - mM)N_z + (2zN_z - nN)M_z = 0,\\
t&: \quad 2zC = 0.
\end{aligned} \tag{14}
$$

As was indicated in §2 the procedure of analysis is to first consider $N_z=0$, $M_z\neq 0$; then $M_z=0$, $N_z\neq 0$; and finally neither equal to zero. If $N_z=0$ and $M_z=0$, we have of course the trivial case. From (14) it is clear that $N_z=0$ unless $2n-1=X$ where X denotes any exponent of t in the system (14). Checking all these possibilities and using (12), we obtain the following property.

PROPERTY 1. *$N_z=0$ for all values of m and n except $n=1$ and $n=2$.*

If $N_z=0$, the system (14) reduces to

$$
\begin{aligned}
t^{2m-1}&: \quad mMM_z = 0,\\
t^{m-1}&: \quad (m-1)M_z = 0, \qquad & t^{2m+1}&: \quad (2zM_z - mM)M_z = 0,\\
t^{m+1}&: \quad (2zB - m)M_z + 2zM_{zz} = 0, & t^{n+m-1}&: \quad nNM_z = 0,\\
t^{n+m+1}&: \quad (2zN_z - n)NM_z = 0, & t&: \quad 2zC = 0.
\end{aligned} \tag{15}
$$

If $M_z=0$, the system (14) reduces to

$$
\begin{aligned}
t^{2n-1}&: \quad nNN_z = 0,\\
t^{n-1}&: \quad (n-1)N_z = 0, \qquad & t^{2n+1}&: \quad (2zN_z - nN)N_z = 0,\\
t^{n+1}&: \quad (2zB - n)N_z = 0, & t^{n+m-1}&: \quad mMN_z = 0,\\
t^{n+m+1}&: \quad (2zM_z - mM)N_z = 0, & t&: \quad 2zC = 0.
\end{aligned} \tag{16}
$$

THEOREM 3. *Case* II *reduces to the trivial case*

(i) *if $n<0$ and $m\geqq 0$;*

(ii) *if* $n<0$ *and* $m<0$;
(iii) *if* $m>2$;
(iv) *if* $n>2$ *and* $m=0$;
(v) *if* $n>3$ *and* $m=1$;
(vi) *if* $n>4$ *and* $m=2$;
(vii) *if* $n=3$ *and* $m=2$.

PROOF. By definition of the trivial case it is necessary to show only that under these conditions $N_s=0$ and $M_s=0$.

(i) and (ii). If $n<0$, $N_s=0$ by Property 1. The consideration then reduces to an analysis of the exponents of t in (15). In (15) $M_s=0$ unless $2m-1=X$ and $n+m-1=X$ simultaneously, X denoting any other exponent of t in the system (15). If $n<0$ and $m\geqq 0$, $2m-1=X$ only if $m=0$, $m=1$, or $m=2$; $n+m-1=X$ only if $m>2$. By a comparison it is easily seen that none of these conditions are satisfied simultaneously, therefore M_s is also zero. If $n<0$ and $m<0$, $n+m-1=X$ only if $n=m+2$; $2m-1=X$ only if $n=m-2$. These conditions can not be satisfied simultaneously, therefore $M_s=0$.

(iii). In (14) $M_s=0$ unless $m-1=X$; but by (i), (ii), and (12) this is impossible for $m>2$; thus $M_s=0$. If $M_s=0$, the system (14) reduces to (16) and $N_s=0$ unless $2n-1=X$, $n-1=X$, and $n+m-1=X$ simultaneously in (16). Checking these conditions one easily sees that this is impossible if $m>2$. Thus N_s is also equal to zero.

We now turn to (iv), (v), and (vi). If $n>2$, $N_s=0$ by Property 1. The system (14) then reduces to (15).

(iv). If $m=0$, the exponents of t in (15) become

$$-1,\ 1,\ n-1,\quad \text{and}\quad n+1.$$

Further $M_s=0$ unless $-1=X$ and $n-1=X$ simultaneously. This is impossible if $n>2$. Therefore M_s is also equal to zero.

(v). If $m=1$, the exponents of t in (15) become

$$1,\ 3,\ 2,\ n,\quad \text{and}\quad n+2.$$

Further $M_s=0$ unless $n=X$ which is impossible if $n>3$. Therefore M_s is also zero.

(vi). If $m=2$, the exponents of t in (15) become

$$3,\ 1,\ 5,\ n+1,\ n+3.$$

Further $M_s=0$ unless $n+1=X$ which is impossible if $n>4$. Therefore M_s is also zero.

(vii). If $n=3$ and $m=2$, it is easily seen that the system (14) yields $N_s=0$ and $M_s=0$.

THEOREM 4. *The only possible integral values for m and n which yield solutions different from those of the trivial case are*

(i) $m=0$ *and* $n=1$ *or* $n=2$;

(ii) $m=1$ *and* $n=2$ *or* $n=3$;

(iii) $m=2$ *and* $n=4$.

PROOF. This theorem follows directly from Theorem 3.

The cases of Theorem 5 have been discussed in [7] and solutions for B, C, M, and N were obtained there.

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ON THE EQUATION $\chi\alpha=\gamma\chi+\beta$ OVER AN ALGEBRAIC DIVISION RING

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1. Introduction and notation. The main purpose of this paper is to give necessary and sufficient conditions in order that the equation

$$\chi\alpha = \gamma\chi + \beta \tag{1}$$

have a solution χ over an algebraic division ring. In case a solution exists, it is given explicitly if it is unique; otherwise, a method of obtaining one of the solutions is given. The application of the results to a quaternion algebra is discussed in the final section.

Let R be a division ring algebraic over its separable[1] center F, and λ a commutative indeterminate over R. Using the notation of Ore,[2] a polynomial $a(\lambda)\in R[\lambda]$ of degree n,

$$a(\lambda) = \alpha_n\lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_0, \tag{2}$$

will be called reduced if $\alpha_n=1$. The unique reduced polynomial $m(\lambda)\in F[\lambda]$ of minimum degree for which $m(\alpha)=0$ will be labelled $m_\alpha(\lambda)$. It is apparent that $m_\alpha(\lambda)$ is irreducible over $F[\lambda]$. The ring of all elements of R which commute with α will be denoted by R_α.

The substitution of an element of R for λ in the polynomial (1) is not well defined, as λ commutes with elements of R, whereas the elements of R do not all commute among themselves. However, unilateral substitution is well defined. We shall use the symbol $a^r(\beta)$ to mean that β has been substituted for λ on the right in (2), so that

$$a^r(\beta) = \alpha_n\beta^n + \alpha_{n-1}\beta^{n-1} + \cdots + \alpha_0. \tag{3}$$

Left substitution is defined similarly—as there is a complete duality between left and right substitution in our case, we shall discuss right substitution only. If $a^r(\beta)=0$, β is called a right root of $a(\lambda)$. The notation $a(\lambda)\,|^r b(\lambda)$ is used to mean that $a(\lambda)$ is a right factor of $b(\lambda)$. As is well known, β is a right root of $a(\lambda)$ if and only if $(\lambda-\beta)\,|^r a(\lambda)$.

2. Preliminary lemmas. A division algorithm exists over $R[\lambda]$. The particular case of interest here is given by

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[1] That is, no irreducible polynomial in $F[\lambda]$ has a multiple root in R.

[2] O. Ore, *Theory of noncommutative polynomials*, Ann. of Math. vol. 34 (1933) pp. 481–508.

$$b(\lambda) = q(\lambda)(\lambda - \alpha) + b^r(\alpha). \tag{4}$$

That is, the remainder on dividing a polynomial $b(\lambda)$ on the right by $(\lambda - \alpha)$ is $b^r(\alpha)$. For any two elements $a(\lambda)$, $b(\lambda)$ of $R[\lambda]$, there exists a unique reduced greatest common right divisor and a unique reduced least common left multiple.

The following lemma is true for any ring[3] R. It is frequently proven for special cases.

LEMMA A. *If* $c(\lambda) = a(\lambda)b(\lambda)$, *than* $b^r(\alpha) = 0$ *implies* $c^r(\alpha) = 0$.

To prove this for a general ring, let $a(\lambda) = \sum_{i=0}^{n} \alpha_i \lambda^i$, $b(\lambda) = \sum_{i=0}^{m} \beta_i \lambda^i$: then

$$c(\lambda) = \sum_{i=0}^{n} \alpha_i \left(\sum_{j=0}^{m} \beta_j \lambda^j \right) \lambda^i. \tag{5}$$

From this form, it is apparent that $c^r(\alpha) = 0$ if $b^r(\alpha) = 0$.

In any polynomial ring which possesses a division algorithm, the following lemma holds.

LEMMA B. *If* $c(\lambda) = a(\lambda)b(\lambda)$, *then* $(\lambda - \alpha) \,|^r\, c(\lambda)$ *if and only if* $(\lambda - \alpha) \,|^r\, a(\lambda)b^r(\alpha)$.

From (4), $c(\lambda) = a(\lambda)q(\lambda)(\lambda - \alpha) + a(\lambda)b^r(\alpha)$, and the lemma follows.

Over a division ring, this lemma can be put in the following form.

LEMMA B′. *If* $c(\lambda) = a(\lambda)b(\lambda)$ *and* $\tau = b^r(\alpha) \neq 0$, *then* $a^r(\tau + \tau^{-1}) = 0$ *if and only if* $c^r(\alpha) = 0$.

This result was obtained by Wedderburn,[4] and later by Richardson[5] and, in a more general form, by Ore.[2] Another result of Wedderburn's[4] is the following lemma.

LEMMA C. *If* $a^r(\tau\alpha\tau^{-1}) = 0$ *for all nonzero elements* $\tau \in R$, *then* $m_\alpha(\lambda) \,|\, a(\lambda)$.

The following fundamental theorem was obtained by Wedderburn[4] for division algebras and holds equally well for algebraic division rings.

LEMMA D. *If* $m_\alpha(\lambda)$ *is of degree* n, *then there exist elements* $\alpha_1 (= \alpha)$, $\alpha_2, \cdots, \alpha_n$ *in* R *such that*

[3] See C. C. MacDuffee, *Vectors and matrices* (Carus Mathematical Monographs, No. 7), Mathematical Association of America, 1943, Theorem 36.

[4] J. H. M. Wedderburn, *On division algebras*, Trans. Amer. Math. Soc. vol. 22 (1921) pp. 130–131.

[5] A. R. Richardson, *Equations over a division algebra*, Messenger of Mathematics vol. 57 (1928) pp. 1–6.

$$m_\alpha(\lambda) = (\lambda - \alpha_n)(\lambda - \alpha_{n-1}) \cdots (\lambda - \alpha_1). \tag{6}$$

A particular factorization of $m_\alpha(\lambda)$ is needed in the proof of Theorem 2. To obtain this, we establish the following lemma.

LEMMA D′. *There exist elements* $\sigma_{11}, \sigma_{12}, \cdots, \sigma_{1n} \in R$ *such that, if*

$$\sigma_{i+1\,j} = \sigma_{ij}\alpha - \sigma_{i\,i-1}\alpha\sigma_{i\,i-1}^{-1}\sigma_{ij}, \qquad i, j = 1, 2, \cdots, n-1, \tag{7}$$

where $\sigma_{i0} = 1$ *for all* i, *then* $m_\alpha(\lambda)$ *has the factorization* (6) *for*

$$\alpha_i = \sigma_{i\,i-1}\alpha\sigma_{i\,i-1}^{-1}, \qquad i = 2, 3, \cdots, n. \tag{8}$$

That this is true can be seen inductively. Assume that $\sigma_{11}, \sigma_{12}, \sigma_{13}, \cdots, \sigma_{1\,k-1}$ exist, $k < n$, so that

$$m_\alpha(\lambda) = a_{k+1}(\lambda)(\lambda - \alpha_k)(\lambda - \alpha_{k-1}) \cdots (\lambda - \alpha_1),$$

where $\alpha_1, \alpha_2, \cdots, \alpha_k$ are given by (7) and (8). Let

$$a_i(\lambda) = a_{k+1}(\lambda)(\lambda - \alpha_k)(\lambda - \alpha_{k-1}) \cdots (\lambda - \alpha_i), \quad i = 1, 2, \cdots, k,$$

and

$$b_i(\lambda) = (\lambda - \alpha_k)(\lambda - \alpha_{k-1}) \cdots (\lambda - \alpha_i), \qquad i = 1, 2, \cdots, k.$$

From Lemma C, there must exist an element $\sigma_{1k} \in R$ such that $b_1^r(\sigma_{1k}\alpha\sigma_{1k}^{-1}) \neq 0$. Then $\sigma_{1k}\alpha\sigma_{1k}^{-1} - \alpha_1 \neq 0$, and by Lemma B′, $a_2^r[(\sigma_{1k}\alpha\sigma_{1k}^{-1} - \alpha_1)\sigma_{1k}\alpha\sigma_{1k}^{-1}(\sigma_{1k}\alpha\sigma_{1k}^{-1} - \alpha_1)^{-1}] = 0$ or $a_2^r(\sigma_{2k}\alpha\sigma_{2k}^{-1}) = 0$. Now, as $b_1(\lambda) = b_2(\lambda)(\lambda - \alpha_1)$, and $b_1^r(\sigma_{1k}\alpha\sigma_{1k}^{-1}) \neq 0$, $b_2^r(\sigma_{2k}\alpha\sigma_{2k}^{-1}) \neq 0$ from Lemma B′. Thus $\sigma_{2k}\alpha\sigma_{2k}^{-1} - \alpha_2 \neq 0$, so that $a_3^r[(\sigma_{2k}\alpha\sigma_{2k}^{-1} - \alpha_2)\sigma_{2k}\alpha\sigma_{2k}^{-1}(\sigma_{2k}\alpha\sigma_{2k}^{-1} - \alpha_2)^{-1}] = 0$ or $a_3^r(\sigma_{3k}\alpha\sigma_{3k}^{-1}) = 0$. By induction, $a_i^r(\sigma_{ik}\alpha\sigma_{ik}^{-1}) = 0$, $i = 1, 2, \cdots, k+1$, so that we can select $\alpha_{k+1} = \sigma_{k+1\,k}\alpha\sigma_{k+1\,k}^{-1}$.

It is apparent that $m_\alpha(\tau\alpha\tau^{-1}) = 0$ for all nonzero $\tau \in R$. That all roots of $m_\alpha(\lambda)$ are of this form is given by the following lemma.

LEMMA E. *If* $m_\alpha(\beta) = 0$, *then* β *is a transform of* α.

To prove this, let $m_\alpha(\lambda) = (\lambda - \beta)a(\lambda)$. From Lemma C, there must exist an element $\tau \in R$ such that $a^r(\tau\alpha\tau^{-1}) \neq 0$. Thus, in view of Lemma B′, $\beta = \sigma\alpha\sigma^{-1}$, where $\sigma = a^r(\tau\alpha\tau^{-1})$.

3. Principal theorems. If either α or γ is in F, equation (1) becomes trivial. Therefore we shall assume that both α and γ are not in F. Define $\nu_0 = \beta$, and, in general,

$$\nu_i = \gamma^i\beta + \gamma^{i-1}\beta\alpha + \cdots + \gamma\beta\alpha^{i-1} + \beta\alpha^i, \qquad i = 1, 2, \cdots.$$

Then, if $m(\lambda) = \sum_{i=0}^{n}\mu_i\lambda^i$ is any polynomial in $F[\lambda]$, any χ which is a

solution of (1) is also a solution[6] of

$$\chi\alpha m(\alpha) = \gamma m(\gamma)\chi + \sum_{i=0}^{n} \mu_i\sigma_i. \tag{9}$$

The discussion of (1) is divided quite naturally into two cases. The first case, which is the easier of the two, is for γ not a transform of α. The second case is for γ a transform of α.

Case 1. As α and γ are not transforms of each other, $m_\alpha(\gamma)\neq 0$ in view of Lemma E. Thus, if we let $m(\lambda)$ of (9) be $m_\alpha(\lambda)$, we obtain

$$\chi = -[m_\alpha(\gamma)]^{-1}\gamma^{-1}\left(\sum_{i=0}^{n} \mu_i\sigma_i\right) \tag{10}$$

as the unique solution of (9). A substitution of this value of χ in (1) shows that it is also a solution of (1). As any solution of (1) is also a solution of (9), (10) gives the unique solution of (1). We have thus established the following theorem:

THEOREM 1. *If α and γ are not transforms of each other, then*

$$\chi\alpha = \gamma\chi + \beta$$

has a unique solution. If γ is not zero, this solution is given by (10).

Case 2. The remaining considerations are for $\gamma=\tau\alpha\tau^{-1}$. It is apparent that the methods of Case 1 now fail, as $m_\alpha(\gamma)=0$. Thus a new approach must now be made.

Equation (1) can now be put in the form

$$\chi\alpha = \tau\alpha\tau^{-1}\chi + \beta.$$

This equation has a solution if and only if the equation

$$\tau^{-1}\chi\alpha = \alpha\tau^{-1}\chi + \tau^{-1}\beta$$

has a solution. Therefore we need only consider an equation of the form

$$\chi\alpha = \alpha\chi + \beta. \tag{11}$$

The existence of solutions of this equation is given by the following theorem.

THEOREM 2. *Let α be an element of R not in F with minimum polynomial $m_\alpha(\lambda)=a(\lambda)(\lambda-\alpha)$ and β be a nonzero element of R. Then the equation*

[6] See M. H. Ingraham and H. C. Trimble, *On the matric equation $TA=BT+C$*, Amer. J. Math. vol. 63 (1941) p. 13.

$$\chi\alpha = \alpha\chi + \beta$$

has a solution χ *in* R *if and only if* $a^r(\beta\alpha\beta^{-1}) = 0$.

PROOF. We shall first assume that there exists an element $\chi \in R$ such that (11) is satisfied. Then, as $m_a(\chi\alpha\chi^{-1}) = 0$ and $\chi\alpha \neq \alpha\chi$, we have by Lemma B′ that $a^r([\chi\alpha - \alpha\chi]\alpha[\chi\alpha - \alpha\chi]^{-1}) = 0$. Thus $a^r(\beta\alpha\beta^{-1}) = 0$, and the first part of the theorem is established.

On the other hand, suppose that $a^r(\beta\alpha\beta^{-1}) = 0$. We shall now use the particular factorization of $m_a(\lambda)$ given in Lemma D′. Let the polynomials $b_{ij}(\lambda)$ be defined by

$$b_{ij}(\lambda) = (\lambda - \alpha_i)(\lambda - \alpha_{i-1}) \cdots (\lambda - \alpha_j), \qquad i \geqq j = 1, 2, \cdots, n.$$

Also, let $\beta_1 = \beta$, and recursively,

$$\beta_i = \beta_{i-1}\alpha - \alpha_i\beta_{i-1}, \qquad i = 2, 3, \cdots, n. \tag{12}$$

There must exist an integer k such that $b^r_{k2}(\beta\alpha\beta^{-1}) \neq 0$, $b^r_{k+1\,2}(\beta\alpha\beta^{-1}) = 0$. As in the proof of Lemma D′, the successive application of Lemma B′ yields

$$b^r_{k+1\,i+1}(\beta_i\alpha\beta_i^{-1}) = 0, \qquad i = 1, 2, \cdots, k.$$

The last application gives $b^r_{k+1\,k+1}(\beta_k\alpha\beta_k^{-1}) = 0$, so that $\alpha_{k+1} = \beta_k\alpha\beta_k^{-1}$. From (8), $\alpha_{k+1} = \sigma_{k+1\,k}\alpha\sigma^{-1}_{k+1\,k}$: thus there must exist an element $\delta_k \in R_a$ such that $\beta_k = \sigma_{k+1\,k}\delta_k$. Now let us assume that there exist elements $\delta_j \in R_a$ and an integer m such that

$$\beta_i = \sum_{j=i}^{k} \sigma_{i+1\,j}\delta_j, \qquad i = m, m+1, \cdots, k.$$

Then it follows from (7), (8), and (12) that

$$\beta_{m-1}\alpha - \alpha_m\beta_{m-1} = \sum_{j=m}^{k} (\sigma_{mj}\alpha - \alpha_m\sigma_{mj})\delta_j,$$

so that

$$\left(\beta_{m-1} - \sum_{j=m}^{k} \sigma_{mj}\delta_j\right)\alpha = \alpha_m\left(\beta_{m-1} - \sum_{j=m}^{k} \sigma_{mj}\delta_j\right).$$

As $\alpha_m = \sigma_{m\,m-1}\alpha\sigma^{-1}_{m\,m-1}$, there must exist an element $\delta_{m-1} \in R_a$ such that

$$\beta_{m-1} = \sum_{j=m-1}^{k} \sigma_{mj}\delta_j.$$

By induction,

$$\beta = \sum_{j=1}^{k} \sigma_{1j}\delta_j.$$

From (7),

$$\beta = \sum_{j=1}^{k} \sigma_{1j}\delta_j\alpha - \alpha\sum_{j=1}^{k} \sigma_{1j}\delta_j$$

and thus, for any $\delta \in R_\alpha$,

$$\chi = \sum_{j=1}^{k} \sigma_{1j}\delta_j + \delta \tag{13}$$

is a solution of (11).

4. **Special considerations.** As a special case of Theorem 2, consider R as the ring of quaternions over a formally real field F, $R = F(1, i, j, k)$. If we let $\bar{\alpha}$ denote the conjugate of α, α not in F, then

$$m_\alpha(\lambda) = (\lambda - \bar{\alpha})(\lambda - \alpha).$$

Thus $a(\lambda) = (\lambda - \bar{\alpha})$, and Theorem 2 can be written in the following form.

COROLLARY 1. *If R is a quaternion algebra over a formally real field F and α is an element of R not in F, then*

$$\chi\alpha = \alpha\chi + \beta$$

has a solution if and only if

$$\beta\alpha = \bar{\alpha}\beta. \tag{14}$$

Having obtained one solution of (11) from (13), say χ_1, then all solutions are given by $\chi_1 + \delta$, $\delta \in R_\alpha$. It is observed that (11) cannot have a solution if $\beta \in R_\alpha$—as $a^r(\beta\alpha\beta^{-1}) = a^r(\alpha)$ in this case, and $a^r(\alpha)$ cannot be zero due to the separability of F. However, it is not true that (11) always has a solution if β is not in R_α. A simple example to show this is as follows: let R be the ring of quaternions over a formally real field F. For $\beta = i + j$ and $\alpha = i$, (14) is not satisfied, and thus (11) can have no solution.

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A RECURRENCE FORMULA FOR THE SOLUTIONS OF CERTAIN LINEAR PARTIAL DIFFERENTIAL EQUATIONS

MORRIS MARDEN

1. Introduction. In a number of recent papers, Bergman[1] has developed the theory of operational methods for transforming analytic functions of a complex variable into solutions of the linear partial differential equation

$$(1.1) \qquad L(U) = U_{z\bar{z}} + a(z, \bar{z})U_z + b(z, \bar{z})U_{\bar{z}} + c(z, \bar{z})U = 0,$$

where $z = x + iy$, $\bar{z} = x - iy$,

$$U_z = \frac{1}{2}\left(\frac{\partial U}{\partial x} - i\,\frac{\partial U}{\partial y}\right), \qquad U_{\bar{z}} = \frac{1}{2}\left(\frac{\partial U}{\partial x} + i\,\frac{\partial U}{\partial y}\right),$$

$$U_{z\bar{z}} = \frac{1}{4}\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right) = \frac{\Delta U}{4},$$

and where the coefficients $a(z, \bar{z})$, $b(z, \bar{z})$ and $c(z, \bar{z})$ are analytic functions of both variables z and $\bar{z}$. The equation (1.1) is equivalent to the system of two real equations

$$\Delta U^{(1)} + 2AU_x^{(1)} + 2BU_y^{(1)} + 2CU_x^{(2)} + 2DU_y^{(2)} + 4c_1U^{(1)} - 4c_2U^{(2)} = 0,$$

$$\Delta U^{(2)} - 2CU_x^{(1)} - 2DU_y^{(1)} + 2AU_x^{(2)} + 2BU_y^{(2)} + 4c_2U^{(1)} + 4c_1U^{(2)} = 0,$$

where

$$U = U^{(1)} + iU^{(2)}; \quad 2A = (a + \bar{d}) + (b + \bar{b}); \quad 2B = i[(\bar{a} - a) - (\bar{b} - b)];$$

$$c = c_1 + ic_2; \quad 2D = (a + \bar{d}) - (b + \bar{b}); \quad 2C = i[(a - \bar{d}) + (b - \bar{b})].$$

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[1] S. Bergman, (a) *Zur Theorie der Funktionen, die eine lineare partielle Differentialgleichung befriedigen*, Rec. Math. (Mat. Sbornik) N.S. vol. 44 (1937) pp. 1169–1198; (b) *The approximation of functions satisfying a linear partial differential equation*, Duke Math. J. vol. 6 (1940) pp. 537–561; (c) *Linear operators in the theory of partial differential equations*, Trans. Amer. Math. Soc. vol. 53 (1943) pp. 130–155; (d) *On the solutions of partial differential equations of the fourth order*, to appear later.

Furthermore, if $a=\bar{b}$ and c is real, then $C=D=c_2=0$, and the two differential equations become real and identical.

For equations (1.1), Bergman proved the existence of two functions $E_1(z, \bar{z}, t)$ and $E_2(z, \bar{z}, t)$, called by him "generating functions of the first kind,"[2] with the following properties:

(1) They have the forms

$$E_1(z, \bar{z}, t) = \exp\left(-\int_0^z a(z, \bar{z})dz\right)[1 + z\bar{z}tE_1^*(z, \bar{z}, t)],$$

$$E_2(z, \bar{z}, t) = \exp\left(-\int_0^z b(z, \bar{z})dz\right)[1 + z\bar{z}tE_2^*(z, \bar{z}, t)],$$

where each $E_k^*(z, \bar{z}, t)$ has continuous first partial derivatives in z, $\bar{z}$ and t for $|t| \leqq 1$ and for z and $\bar{z}$ within a certain four-dimensional region.

(2) The classes $C(E_1)$ and $C(E_2)$ of functions $U_1(z, \bar{z})$ and $U_2(z, \bar{z})$ defined by the formulas

$$(1.2) \qquad U_1(z, \bar{z}) = \int_{-1}^{1} E_1(z, \bar{z}, t)f(z(1 - t^2)/2)dt/(1 - t^2)^{1/2},$$

$$(1.3) \qquad U_2(z, \bar{z}) = \int_{-1}^{1} E_2(z, \bar{z}, t)g(\bar{z}(1 - t^2)/2)dt/(1 - t^2)^{1/2},$$

where $f(\zeta)$ and $g(\zeta)$ are arbitrary analytic functions of ζ, form subsets of solutions of (1.1).

(3) Every solution $U(z, \bar{z})$ of (1.1) may be written in the form

$$U(z, \bar{z}) = U_1(z, \bar{z}) + U_2(z, \bar{z}),$$

with $f(\zeta)$ and $g(\zeta)$ suitably chosen analytic functions.

As was proved by Bergman, to many theorems about analytic functions of a complex variable correspond analogous theorems about functions belonging to classes $C(E)$ generated by functions E of the first kind. In particular, if we define as "basic solutions" those corresponding to $f(z)=z^p$, that is

$$(1.4) \qquad u_p(z, \bar{z}) = \int_{-1}^{1} E(z, \bar{z}, t)[z(1 - t^2)/2]^p dt/(1 - t^2)^{-1/2},$$

then every function U of class $C(E)$ which is regular in $|z| \leqq r$ may

[2] Generating functions which are considered as not of the first kind are those failing to satisfy property (1). When E is a generating function not of the first kind, the integration in (1.2) and (1.3) must be taken along a rectifiable curve joining the points $t=\pm 1$, but not passing through $t=0$.

be expanded in a series $U=\sum\alpha_p u_p$ which is uniformly and absolutely convergent in $|z|\leqq r$.

For example, in the case of the equation

$$\Delta U + U = 0, \tag{1.5}$$

$E(z, \bar{z}, t)=e^{izr}$, where $z=re^{i\theta}$ and thus $r=(z\bar{z})^{1/2}$. Because of the well known formula for the Bessel function of the first kind

$$J_p(r) = (2/\pi)(1/\Gamma(p + 1/2)) \int_{-1}^{1} e^{rti}(r/2)^p(1 - t^2)^{p-1/2}dt,$$

the basic solutions are

$$u_p(r, \theta) = (\pi^{1/2}/2)\Gamma(p + 1/2)e^{p\theta i}J_p(r). \tag{1.6}$$

But for (1.5), successive terms in the expansion $U=\sum\alpha_p u_p$ can be computed from earlier terms by the use of some recurrence relation satisfied by the Bessel's functions, as for example the relation

$$J_p'(r) = (p/r)J_p(r) - J_{p+1}(r). \tag{1.7}$$

It would likewise be of practical value in the case of other differential equations $L(U)=0$ to determine what recurrence relations, if any, are satisfied by the basic solutions $u_p(r, \theta)$.

In the present note, recurrence formulas connecting the basic solutions $u_p(r, \theta)$ are found in the case of differential equations $L(U)=0$ for which at least one of the corresponding "generating functions" $E(z, \bar{z}, t)$ is of the form $E(z, \bar{z}, t)=\exp f(r, \theta, t)$ where $f(r, \theta, t)$ is a polynomial in t containing either only even powers of t or only odd powers of t. Obviously, the equation (1.5) is an example of such an equation. Other examples can be found by requiring the coefficients a, b and c in the equation $L(U)=0$ to satisfy certain differential relations.[3]

Our first main result may be stated as follows:

THEOREM 1. *Let $L(U)=0$ be a partial differential equation of the type (1.1) for which there exists a generating function having one of the forms*:

$$\text{(I)} \qquad E(z, \bar{z}, t) = \exp P(r, \theta, t),$$

$$\text{(II)} \qquad E(z, \bar{z}, t) = \exp tP(r, \theta, t),$$

where $P(r, \theta, t)=a_0(r, \theta)+a_1(r, \theta)t^2+\cdots+a_n(r, \theta)t^{2n}$, and where the coefficients $a_j(r, \theta)$ are of class C' in r and θ. Let $u_p(r, \theta)$ be the corresponding "basic solutions" of equation $L(U)=0$ and let

[3] See reference in footnote 1a, pp. 1194–1195, and also p. 158 of the following article: K. L. Nielsen and B. P. Ramsay, *On particular solutions of linear partial differential equations*, Bull. Amer. Math. Soc. vol. 49 (1943) pp. 156–162.

$$\alpha_q(r, \theta) = (-1)^q \left[\sum_{j=q}^{n} C_{j,q} \frac{\partial a_j}{\partial r} \right] \left(\frac{2}{re^{i\theta}} \right)^q, \tag{1.8}$$

$$\beta_q(r, \theta) = (-1)^q \left[\sum_{j=q}^{n} (2j+1) C_{j,q} a_j \right] \left(\frac{2}{re^{i\theta}} \right)^q. \tag{1.9}$$

Then, if E has form (I),

$$\frac{\partial u_p}{\partial r} = \frac{p}{r} u_p + \sum_{j=0}^{n} \alpha_j u_{p+j}; \tag{1.10}$$

whereas, if E has form (II),

$$\frac{\partial u_p}{\partial r} = \frac{p}{r} u_p + \frac{2}{re^{i\theta}} \sum_{j,k=0}^{n} \frac{\alpha_j \beta_k u_{j+k+p+1}}{2j + 2p + 1}. \tag{1.11}$$

The above theorem will be derived as an immediate consequence of two lemmas that are given in the next section. In the third section the theorem will be applied to a few specific equations of form (1.1).

2. **Two lemmas.** First we shall derive a result for polynomials $P(r, \theta, t)$ involving only even powers of t.

LEMMA 1. *Let*

$$P(r, \theta, t) = a_0(r, \theta) + a_1(r, \theta) t^2 + \cdots + a_n(r, \theta) t^{2n},$$

where the $a_j(r, \theta)$ are functions of class C' in r and θ. Then the function

$$u_p(r, \theta) = \int_{-1}^{1} e^{P(r,\theta,t)} (1 - t^2)^{p-1/2} (re^{i\theta}/2)^p dt \tag{2.1}$$

satisfies the recurrence formula

$$\partial u_p / \partial r = [p/r + P_r(r, \theta, (1 - T)^{1/2})] u_p, \tag{2.2}$$

where T is the operator such that, T acting k times upon u_p,

$$T^k u_p = (2/re^{i\theta})^k u_{p+k} \tag{2.3}$$

for $k = 0, 1, \cdots$.

PROOF. From (2.1) we obtain by differentiating with respect to r:

$$\frac{\partial u_p}{\partial r} = \frac{p}{r} u_p + \int_{-1}^{1} e^P \left[\frac{\partial a_0}{\partial r} + \frac{\partial a_1}{\partial r} t^2 + \cdots + \frac{\partial a_n}{\partial r} t^{2n} \right] (1 - t^2)^{p-1/2} \left(\frac{re^{i\theta}}{2} \right) dt. \tag{2.4}$$

To evaluate the latter integral, let us note that

$$\begin{aligned}\int_{-1}^{1} t^{2m}e^{p}(1-t^2)^{p-1/2}(re^{i\theta}/2)^p dt &= \int_{-1}^{1}[1-(1-t^2)]^m e^{p}(1-t^2)^{p-1/2}(re^{i\theta}/2)^p dt \\ &= \sum_{k=0}^{m}(-1)^k C_{m,k}\int_{-1}^{1} e^{p}(1-t^2)^{k+p-1/2}(re^{i\theta}/2)^p dt \\ &= \sum_{k=0}^{m}(-1)^k C_{m,k}(2/re^{i\theta})^p u_{p+k}.\end{aligned}$$

Hence,

$$\int_{-1}^{1} t^{2m}e^{p}(1-t^2)^{p-1/2}(re^{i\theta}/2)^p dt = (1-T)^m u_p. \tag{2.5}$$

Substituting now from (2.5) into (2.4), we find

$$\begin{aligned}\frac{\partial u_p}{\partial r} &= \frac{p}{r}u_p + \frac{\partial a_0}{\partial r}u_p + \frac{\partial a_1}{\partial r}(1-T)u_p + \cdots + \frac{\partial a_n}{\partial r}(1-T)^n u_p \\ &= [p/r + P_r(r,\theta,(1-T)^{1/2})]u_p,\end{aligned}$$

as was to be proved.

The corresponding result for a polynomial that involves only odd powers of t may be stated as follows.

LEMMA 2. *Let* $Q(r,\theta,t) \equiv a_0(r,\theta)t + a_1(r,\theta)t^3 + \cdots + a_n(r,\theta)t^{2n+1} \equiv tP(r,\theta,t)$, *where the* $a_i(r,\theta)$ *are functians of class* C' *in* r *and* θ. *Then the functions* $u_p(r,\theta)$ *defined by* (2.1) *satisfy the recurrence formula*:

$$\begin{aligned}\partial u_p/\partial r = (p/r)u_p + \{&Q_t(r,\theta,(1-T)^{1/2}) \\ &\cdot \int_0^{T^{1/2}} t^{2p}P_r(r,\theta,(1-t)^{1/2})dt\Big\}T^{-p+1/2}u_p\,,\end{aligned} \tag{2.6}$$

where T *is the operator defined by equation* (2.3).

PROOF. In place of (2.4), we now have

$$\begin{aligned}\frac{\partial u_p}{\partial r} = \frac{p}{r}u_p + \int_{-1}^{1} e^{Q}\Big[&\frac{\partial a_0}{\partial r}t + \frac{\partial a_1}{\partial r}t^3 + \cdots \\ &+ \frac{\partial a_n}{\partial r}t^{2n+1}\Big](1-t^2)^{p-1/2}\left(\frac{re^{i\theta}}{2}\right)^p dt.\end{aligned} \tag{2.7}$$

In order to evaluate the latter integral, let us first integrate by parts:

$$\int_{-1}^{1} te^{Q}(1-t^2)^{p-1/2}(re^{i\theta}/2)^p dt$$

$$= -\frac{1}{2}\int_{-1}^{1} e^{Q}(1-t^2)^{p-1/2}(re^{i\theta}/2)^p d(1-t^2)$$

$$= (1/(2p+1))\int_{-1}^{1} e^{Q}(\partial Q/\partial t)(1-t^2)^{p+1/2}(re^{i\theta}/2)^p dt$$

$$= (1/(2p+1))\int_{-1}^{1} e^{Q}[a_0 + 3a_1t^2 + \cdots + (2n+1)a_nt^{2n}](1-t^2)^{p+1/2}(re^{i\theta}/2)^p dt.$$

Hence, by equation (2.5),

$$\int_{-1}^{1} te^{Q}(1-t^2)^{p-1/2}(re^{i\theta}/2)^p dt = (1/(2p+1))Q_i(r,\theta,(1-T)^{1/2})Tu_p$$

$$= Q_i(r,\theta,(1-T)^{1/2})\left(\int_0^{T^{1/2}} t^{2p}dt\right)T^{-p+1/2}u_p.$$

Let us then assume that the formula

$$\int_{-1}^{1} t^{2m+1}e^{Q}(1-t^2)^{p-1/2}(re^{i\theta}/2)^p dt$$

$$= Q_i(r,\theta,(1-T)^{1/2})\left(\int_0^{T^{1/2}} t^{2p}(1-t^2)^m dt\right)T^{-p+1/2}u^p \tag{2.8}$$

has already been verified for $m=0, 1, 2, \cdots, N$ and proceed to verify the formula for $m=N+1$, as follows.

$$\int_{-1}^{1} t^{2N+3}e^{Q}(1-t^2)^{p-1/2}(re^{i\theta}/2)^p dt$$

$$= \int_{-1}^{1} t^{2N+1}e^{Q}(1-t^2)^{p-1/2}(re^{i\theta}/2)^p dt$$

$$- \int_{-1}^{1} t^{2N+1}e^{Q}(1-t^2)^{p+1/2}(re^{i\theta}/2)^p dt$$

$$= Q_i(r,\theta,(1-T)^{1/2})\left\{\left[\int_0^{T^{1/2}} t^{2p}(1-t^2)^N dt\right]T^{-p+1/2}u_p\right.$$

$$\left.- \left[\int_0^{T^{1/2}} t^{2p+2}(1-t^2)^N dt\right](2/re^{i\theta})T^{-p-1/2}u_{p+1}\right\}$$

$$= Q_i(r,\theta,(1-T)^{1/2})\left(\int_0^{T^{1/2}} t^{2p}(1-t^2)^{N+1}dt\right)T^{-p+1/2}u_p.$$

Thus, formula (2.8) has been established by mathematical induction.

We now substitute from formula (2.8) into expression (2.7), thus obtaining

$$\frac{\partial u_p}{\partial r}=\frac{\hat{p}}{r}u_p+\Bigg(Q_i(r,\theta,(1-T)^{1/2})\cdot\int_0^{T^{1/2}}t^{2p}\left\{\frac{\partial a_0}{\partial r}+\frac{\partial a_1}{\partial r}(1-t^2)+\cdots+\frac{\partial a_n}{\partial r}(1-t^2)^n\right\}\Bigg)T^{-p+1/2}u_p$$

$$=\frac{\hat{p}}{r}u_p+Q_i(r,\theta,(1-T)^{1/2})\left\{\int_0^{T^{1/2}}t^{2p}P_r(r,\theta,(1-t)^{1/2})\right\}T^{-p+1/2}u_p,$$

as was to be proved.

PROOF OF THEOREM 1. Formula (2.2) may be reduced to formula (1.10) if, using (1.8) and (2.3), we set

$$P_r(r,\theta,(1-T)^{1/2})u_p=\sum_{j=0}^{n}\frac{\partial a_j}{\partial r}(1-T)^j u_p=\sum_{q=0}^{n}\alpha_q\left(\frac{re^{i\theta}}{2}\right)^q T^q u_p$$

$$=\sum_{q=0}^{n}\alpha_q u_{p+q}.$$

To reduce formula (2.6) to (1.11), let us set

$$Q_i(r,\theta,(1-T)^{1/2})=\sum_{j=0}^{n}(2j+1)a_j(1-T)^j=\sum_{j=0}^{n}\beta_j\left(\frac{re^{i\theta}}{2}\right)^j T^j$$

with the β_j defined as in formula (1.9). Since

$$P_r(r,\theta,(1-t^2)^{1/2})=\sum_{j=0}^{n}\alpha_j\left(\frac{re^{i\theta}}{2}\right)^j t^{2j},$$

we may write the second term of the left side of (2.6) as

$$\sum_{k=0}^{n}\beta_k\left(\frac{re^{i\theta}}{2}\right)^k T^k\left[\int_0^{T^{1/2}}t^{2p}\sum_{j=0}^{n}\alpha_j\left(\frac{re^{i\theta}}{2}\right)^j t^{2j}dt\right]T^{-p+1/2}u_p$$

$$=\sum_{j,k=0}^{n}\frac{\alpha_j\beta_k}{2p+2j+1}\left(\frac{re^{i\theta}}{2}\right)^{j+k}T^{k+j+1}u_p=\frac{2}{re^{i\theta}}\sum_{j,k=0}^{n}\frac{\alpha_j\beta_k u_{p+j+k+1}}{2p+2j+1}.$$

Thus the proof of our main theorem is completed.

3. **Examples.** Let us first verify that the recurrence relation (2.6) is a generalization of that for Bessel's functions as given in formula (1.7). Here $Q(r,\theta,t)=rti$ and thus (2.6) becomes

$$\partial u_p/\partial r = (p/r)u_p + ri\left[\int_0^{r^{1/2}} t^{2p} i dt\right] T^{-p+1/2}u_p$$

(3.1)

$$= (p/r)u_p - (2e^{-i\theta}/(2p+1))u_{p+1}.$$

If now we set

$$u_p = (\pi^{1/2}/2)\Gamma(p+1/2)J_p(r)e^{ip\theta},$$
$$u_{p+1} = (\pi^{1/2}/2)(p+1/2)\Gamma(p+1/2)J_{p+1}(r)e^{i(p+1)\theta},$$

formula (3.1) reduces at once to formula (1.7).

As our second example, let us consider the differential equation $L(U)=0$ in which the expression $F=c-ab-a_z\neq 0$ satisfies the two equations

$$F_z = 0, \qquad 2F - a_r + b_{\bar{z}} = 0. \tag{3.2}$$

As shown by Bergman,[4] one of the possible corresponding generating functions is $E(z, \bar{z}, t)=\exp P(r, \theta, t)=\exp(a_0+a_1t^2)$, where

$$a_0 = -\int_0^z a d\bar{z}, \qquad a_1 = 2z\int_0^z F d\bar{z}. \tag{3.3}$$

According to our theorem, the recurrence relation satisfied by the basic solutions is in this case

$$(\partial u_p/\partial r) = (p/r)u_p + \alpha_0 u_p + \alpha_1 u_{p+1}, \tag{3.4}$$

where

$$\alpha_0 = \partial a_0/\partial r + \partial a_1/\partial r = -ae^{-i\theta} + 2rF + 2e^{i\theta}\int_0^z F d\bar{z},$$

$$\alpha_1 = -(2/re^{i\theta})(\partial a_1/\partial r) = -4e^{-i\theta}F - (4/r)\int_0^z F d\bar{z}.$$

A partial differential equation which satisfies conditions (3.2) is

$$U_{z\bar{z}} - 2(z+\bar{z})U_{\bar{z}} + U = 0. \tag{3.5}$$

Here $F(z)=1$, $a_0=0$, $a_1=2r^2$, and therefore in the recurrence relation (3.4) $\alpha_0=4r$, and $\alpha_1=-8e^{-i\theta}$. Setting $U=U^{(1)}+iU^{(2)}$, we see that equation (3.5) is equivalent to the system of two partial differential equations

$$\Delta U^{(1)} - 8xU_x^{(1)} + 8xU_y^{(2)} + 4U^{(1)} = 0,$$
$$\Delta U^{(2)} - 8xU_x^{(2)} - 8xU_y^{(1)} + 4U^{(2)} = 0,$$

[4] See p. 1194, reference in footnote 1a.

and that recurrence relation (3.4) for $u_p = u_p^{(1)} + iu_p^{(2)}$ is in this case equivalent to the system of recurrence relations

$$\partial u_p^{(1)}/\partial r = (p/r)u_p^{(1)} + 4ru_p^{(1)} - 8u_{p+1}^{(1)} \cos\theta - 8u_{p+1}^{(2)} \sin\theta,$$
$$\partial u_p^{(2)}/\partial r = (p/r)u_p^{(2)} + 4ru_p^{(2)} - 8u_{p+1}^{(2)} \cos\theta + 8u_{p+1}^{(1)} \sin\theta.$$

Other examples of partial differential equations $L(U)=0$ for which $\log E$ is an even or odd polynomial in t may be found in the articles referred to in footnotes 1a and 3. For these differential equations also, a recurrence relation may be derived by use of Lemmas 1 and 2.

4. **Generalization.** By means of formulas (2.5) and (2.8), the theorem given in the introduction may be extended to partial differential equations of type (1.1) for which a generating function exists that has the form $E=g \exp f$ with both f and g suitably chosen polynomials in t. The generalization may be stated as follows.

THEOREM 2. *Let $L(U)=0$ be a partial differential equation of type (1.1) for which a generating function $E(z, \bar{z}, t)$ exists that has one of the forms*

$$\text{I.} \quad E(z, \bar{z}, t) = R(r, \theta, t) \exp P(r, \theta, t),$$
$$\text{II.} \quad E(z, \bar{z}, t) = R(r, \theta, t) \exp tP(r, \theta, t),$$
$$\text{III.} \quad E(z, \bar{z}, t) = tR(r, \theta, t) \exp tP(r, \theta, t),$$

where

$$P(r, \theta, t) = a_0(r, \theta) + a_1(r, \theta)t^2 + \cdots + a_m(r, \theta)t^{2m},$$
$$R(r, \theta, t) = b_0(r, \theta) + b_1(r, \theta)t^2 + \cdots + b_n(r, \theta)t^{2n},$$

and where the $a_j(r, \theta)$ and $b_j(r, \theta)$ are of class C' in r and θ. Let $u_p(r, \theta)$ be the corresponding basic solutions and let $R(\partial p/\partial r) = \sum_0^{m+n} c_j(r, \theta)t^{2j}$,

$$\alpha_k = (-1)^k \left(\frac{2}{re^{i\theta}}\right)^k \sum_{j=k}^{m} (2j+1)C_{j,k}a_j;$$

$$\beta_k = (-1)^k \left(\frac{2}{re^{i\theta}}\right)^k \sum_{j=k}^{n} C_{j,k} \frac{\partial b_j}{\partial r}; \quad \gamma_k = (-1)^k \left(\frac{2}{re^{i\theta}}\right)^k \sum_{j=k}^{n+m} C_{j,k}c_j.$$

Then the recurrence relation satisfied by these basic solutions is

$$\frac{\partial u_p}{\partial r} = \frac{pu_p}{r} + \sum_{k=0}^{n} \beta_k u_{p+k} + \sum_{k=0}^{n+m} \gamma_k u_{p+k}$$

if E has the form I;

$$\frac{\partial u_p}{\partial r} = \frac{pu_p}{r} + \sum_{k=0}^{n} \beta_k u_{p+k} + \left(\frac{2}{re^{i\theta}}\right) \sum_{k=0}^{m} \sum_{\nu=0}^{m+n} \frac{\alpha_k \gamma_\nu u_{p+k+\nu+1}}{2p+2\nu+1}$$

if E has the form II; *and*

$$\frac{\partial u_p}{\partial r} = \frac{pu_p}{r} + \left(\frac{2}{re^{i\theta}}\right) \sum_{k=0}^{m} \sum_{\nu=0}^{n} \left(\frac{\alpha_k \beta_\nu}{2p+2\nu+1}\right) u_{p+k+\nu+1} + \gamma_0 u_p + \sum_{k=1}^{m+n} \left(\gamma_k - \frac{2}{re^{i\theta}} \gamma_{k-1}\right) u_{p+k}$$

if E has the form III.

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ON GAUSS' AND TCHEBYCHEFF'S QUADRATURE FORMULAS

J. GERONIMUS

The well known Gauss' Quadrature Formula

$$\int_{-\infty}^{\infty} G_k(x)d\psi(x) = \sum_{i=1}^{n} \rho_i^{(n)} G_k(\xi_i^{(n)}) \tag{1}$$

is valid for every polynomial $G_k(x)$, of degree $k \leqq 2n-1$, the $\{\xi_i^{(n)}\}$ being the roots of the polynomial $P_n(x)$, orthogonal with respect to the distribution $d\psi(x)$ $(i=1, 2, \cdots, n;\ n=1, 2, \cdots)$.[1] If the sequence $\{P_n(x)\}$ is that of Tchebycheff (trigonometric) polynomials, then the Christoffel numbers $\rho_i^{(n)}$, $i=1, 2, \cdots, n$, are equal, and the two quadrature formulas of Gauss and Tchebycheff coincide:

$$\int_{-\infty}^{\infty} G_k(x)d\psi(x) = \rho_n \sum_{i=1}^{n} G_k(\xi_i^{(n)}), \qquad k \leqq 2n-1;\ n = 1, 2, \cdots. \tag{2}$$

The converse—that this is the only case of coincidence of these formulas—was proved by R. P. Bailey [1a] and, under more restrictive conditions, by Krawtchouk [1b] (cf. also [2]).[2]

We shall give here four distinct proofs of this statement, without imposing any restrictions on $\psi(x)$.

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[1] $\psi(x)$ is a bounded non-decreasing function, with infinitely many points of increase, for which all moments exist: $c_n = \int_{-\infty}^{\infty} x^n d\psi(x)$; $n=0, 1, 2, \cdots$.

[2] Numbers in brackets refer to the bibliography at the end of the paper.

Consider the sequence of complex numbers $\{c_n\}$, $n=0, 1, 2, \cdots$, subject to the conditions

$$\Delta_{n+1} \equiv |c_{i+k}|_{i,k=0}^{n} \neq 0; \qquad n = 0, 1, 2, \cdots. \tag{3}$$

Consider also the Stieltjes linear functional σ, with

$$\sigma(x^k) = c_k; \qquad k = 0, 1, 2, \cdots, \tag{4}$$

and the polynomials $\{P_n(x)\}$, $n=0, 1, 2, \cdots$, orthogonal relative to the sequence $\{c_n\}$, that is [3, 4]

$$\sigma\{P_n(x)P_m(x)\} = \begin{cases} 0, & m \neq n, \\ h_n \equiv \Delta_{n+1}/\Delta_n \neq 0, & m = n. \end{cases} \tag{5}$$

Let $\{\xi_i^{(n)}\}$, $i=1, 2, \cdots, n$, be the roots, distinct or not, of $P_n(x)$, $n=1, 2, \cdots$.

I. The first method of proving our statement consists in proving the following theorem.

THEOREM. *From the validity of the formula*

$$\rho_n \sum_{i=1}^{n} (\xi_i^{(n)})^k = c_k, \qquad n \geqq \left[\frac{k+1}{2}\right], \tag{6}$$

for $k \leqq 2$, it follows that the $\{P_n(x)\}$ are the Tchebycheff polynomials, so that (6) holds for all integral k.

Introduce the mean of order ν of the numbers $\{\xi_i^{(n)}\}$, $i=1, 2, \cdots, n$:

$$\mu_\nu^{(n)} = \left[\frac{1}{n}\sum_{i=1}^{n} (\xi_i^{(n)})^\nu\right]^{1/\nu}, \qquad \nu = 1, 2, \cdots; n = 1, 2, \cdots. \tag{7}$$

On equating the coefficients of x^{n-1} and x^{n-2} on both sides of the recurrence relation

$$P_n(x) = (x - \alpha_n)P_{n-1}(x) - \lambda_n P_{n-2}(x), \qquad \lambda_n \neq 0, \quad P_{-1} \equiv 0, \; n = 1, 2, \cdots, \tag{8}$$

we get

$$\delta_1^{(n)} = -\sum_{k=1}^{n} \alpha_k, \qquad \delta_2^{(n)} = -\sum_{k=2}^{n} \lambda_k + \sum_{r=2}^{n} \alpha_r \sum_{k=1}^{r-1} \alpha_k, \tag{9}$$

where we let

$$P_n(x) = x^n + \delta_1^{(n)} x^{n-1} + \delta_2^{(n)} x^{n-2} + \cdots + \delta_n^{(n)}. \tag{9'}$$

On putting $k=0, 1, 2$ in (6) we obtain

$$\rho_n = c_0/n; \quad \mu_1^{(n)} = c_1/c_0, \ n \geqq 1; \quad \mu_2^{(n)} = c_2/c_0, \ n \geqq 2. \tag{10}$$

Since, by (9) and (9′),

$$\begin{aligned} n\mu_1^{(n)} &= -\delta_1^{(n)} = \sum_{k=1}^{n} \alpha_k, \\ n(\mu_2^{(n)})^2 &= (\delta_1^{(n)})^2 - 2\delta_2^{(n)} = \Big(\sum_{k=1}^{n} \alpha_k\Big)^2 + 2\sum_{k=1}^{n} \lambda_k - 2\sum_{r=2}^{n}\sum_{k=1}^{r-1} \alpha_k, \end{aligned} \tag{11}$$

we find:

$$\alpha_1 = (\alpha_1 + \alpha_2)/2 = (\alpha_1 + \alpha_2 + \alpha_3)/3 = \cdots,$$

whence we have the fundamental result:

$$\alpha_1 = \alpha_2 = \cdots = \alpha, \tag{12}$$

and similarly,

$$\lambda_3 = \lambda_4 = \cdots = \lambda_2/2 = \lambda. \tag{12′}$$

Here α and λ are independent of n. The solution of (8) under these conditions is

$$\begin{aligned} P_n(x) = {} & [(x - \alpha + ((x-\alpha)^2 - 4\lambda)^{1/2})^n \\ & + (x - \alpha - ((x-\alpha)^2 - 4\lambda)^{1/2})^n]/2^n, \quad n = 1, 2, \cdots, \end{aligned} \tag{13}$$

which shows that the $\{P_n(x)\}$ are the Tchebycheff polynomials, whence the validity of (6) for $k=3, 4, \cdots$ follows.

We have found incidentally the following property of the Tchebycheff polynomials.

COROLLARY. *If the arithmetic mean and the root-mean-square of the roots* $\{\xi_i^{(n)}\}$, $i=1, 2, \cdots, n$, *of the orthogonal polynomials* $P_n(x)$, $n=1, 2, \cdots$, *do not depend on* n, *the means of all orders* $\nu \leqq 2n-1$ *possess the same property, and the* $\{P_n(x)\}$ *are the Tchebycheff polynomials* (13).

II. Define

$$R_{n-1}(y) = \sigma\{(P_n(y) - P_n(x))/(y-x)\},$$

a polynomial of degree $n-1$; $n=1, 2, \cdots$. Our second proof consists in showing that (6) implies

$$R_{n-1}(y) \equiv (c_0/n)P_n'(y), \qquad n \geqq 2. \tag{14}$$

From (6), wherein ρ_n necessarily equals c_0/n, we get, by virtue of (4),

$$\sigma(x^k) = \frac{c_0}{n}\sum_{i=1}^{n}(\xi_i^{(n)})^k; \quad \sigma(G_k(x)) = \frac{c_0}{n}\sum_{i=1}^{n}G_k(\xi_i^{(n)}), \qquad n \geqq \left[\frac{k+1}{2}\right].$$

The desired relation

$$R_{n-1}(y) = \frac{c_0}{n}\sum_{i=1}^{n}\frac{P_n(y) - P_n(\xi_i^{(n)})}{y - \xi_i^{(n)}} = \frac{c_0}{n}P_n(y)\sum_{i=1}^{n}\frac{1}{y - \xi_i^{(n)}} \equiv \frac{c_0}{n}P_n'(y)$$

follows. Now there are different methods of showing that (14) implies (13).

II_1. This statement is proved in a note [2] as a particular case of more general theorems.

II_2. Introduce the polynomials $\{P_n^{(i)}(x)\}$, $i=1, 2, \cdots$, n—the denominators of the convergents of order n of the continued fractions[3]

$$\frac{\lambda_i|}{|x - \alpha_i} - \frac{\lambda_{i+1}|}{|x - \alpha_{i+1}} - \cdots, \qquad i = 1, 2, \cdots. \tag{15}$$

We have $P_n^{(1)}(x) \equiv P_n(x)$; $P_n^{(2)}(x) \equiv (1/c_0)R_n(x)$, $n=0, 1, 2, \cdots$.

It is easy to show that

$$P_n(x) = (x - \alpha_1)P_{n-1}^{(2)}(x) - \lambda_2 P_{n-2}^{(3)}(x), \qquad n = 2, 3, \cdots. \tag{16}$$

On the other hand

$$R_{n-1}(x) = (x - \alpha_n)R_{n-2}(x) - \lambda_n R_{n-3}(x), \qquad n = 3, 4, \cdots. \tag{17}$$

Using (8), (14) and (17), we find that

$$P_{n-1}(x) = (x - \alpha_n)P_{n-2}^{(2)}(x) - 2\lambda_n P_{n-3}^{(2)}(x), \qquad n = 3, 4, \cdots, \tag{18}$$

which gives, in conjunction with (16),

$$(\alpha_1 - \alpha_n)P_{n-2}^{(2)}(x) = 2\lambda_n P_{n-3}^{(2)}(x) - \lambda_2 P_{n-3}^{(3)}(x), \qquad n = 3, 4, \cdots. \tag{19}$$

Hence

$$\alpha_3 = \alpha_4 = \cdots = \alpha_1 = \alpha; \qquad \lambda_3 = \lambda_4 = \cdots = \lambda_2/2 = \lambda; \qquad P_{n-3}^{(2)}(x) \equiv P_{n-3}^{(3)}(x). \tag{20}$$

This identity, for $n=4$, gives $\alpha_2=\alpha$, and thus we have arrived again at (12, 12′).

II_3. On putting

[3] They were introduced by Stieltjes [5]; cf. also Perron [6].

$$R_n(y) = c_0(y^n + d_1^{(n)} y^{n-1} + \cdots + d_n^{(n)}), \tag{21}$$

we find from (17) the relations

$$d_1^{(n-1)} = -\sum_{k=2}^{n} \alpha_k, \qquad d_2^{(n-1)} = \sum_{k=3}^{n} \alpha_k \sum_{r=2}^{k-1} \alpha_r - \sum_{k=3}^{n} \lambda_k \tag{22}$$

analogous to (9). From (14) we find

$$nd_1^{(n-1)} = (n-1)b_1^{(n)}, \tag{23}$$

whence again $\alpha_1=\alpha_2= \cdots =\alpha$; and in the same way, the condition

$$nd_2^{(n-1)} = (n-2)b_2^{(n)} \tag{24}$$

again implies $\lambda_3=\lambda_4= \cdots =\lambda_2/2=\lambda$.

We have found incidentally the following property of the Tchebycheff polynomials.

COROLLARY. *If the three highest coefficients of the polynomials* $\{nR_{n-1}(x)\}$ *and* $\{c_0P_n'(x)\}$, $n=1, 2, \cdots$, *coincide, then these polynomials are identical, and the* $\{P_n(x)\}$ *are the Tchebycheff polynomials* (13).

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POWERS OF HOMEOMORPHISMS WITH ALMOST PERIODIC PROPERTIES

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Let X be a topological space (an "accessible space," a "1-space," or a "T_1-space" in the terminology of Fréchet, Kuratowski, or Alexandroff-Hopf, respectively) and let $f(X)=X$ be a homeomorphism. We use the following terminology, which was suggested by G. A. Hedlund and which is to be carefully distinguished from those terminologies used by Birkhoff, Ayres, Whyburn, and others. A point x of X is said to be *recurrent* under f provided that to each neighborhood U of x there corresponds a positive integer n such that $f^n(x)\in U$. The mapping f is said to be *pointwise recurrent* provided that each point of X is recurrent under f. A point x of X is said to be *almost periodic* under f provided that to each neighborhood U of x there corresponds a monotone increasing sequence $n_1, n_2, \cdots$ of positive integers with the properties that the numbers $n_{i+1}-n_i$ $(i=1, 2, \cdots)$ are uniformly bounded and $f^{n_i}(x)\in U$ $(i=1, 2, \cdots)$. The mapping f is said to be *pointwise almost periodic* provided each point of X is almost periodic under f. Following Birkhoff [1, p. 198],[2] a subset Y of X is said to be *minimal* under f provided that Y is nonvacuous, closed and invariant under f, that is, $f(Y)=Y$, and furthermore Y does not contain a proper subset with these properties. For $x\in X$, the set $n\sum_{-\infty}^{+\infty}f^n(x)$ is called the *orbit* of x under f and the set $\sum_{n=0}^{+\infty}f^n(x)$ is called the *semi-orbit* of x under f. A *decomposition* of X is a collection of nonvacuous pairwise disjoint closed subsets of X which fill up X.

THEOREM 1. *If $x\in X$ is recurrent under f, then x is also recurrent under f^n for every positive integer n.*

PROOF. We make use of an induction. The theorem is true for $n=1$. Let m be any positive integer. Assume the theorem is true for $n\leqq m$. We now show the theorem is true for $n=m+1=k$.

We may suppose without loss of generality that X is the closure of the semi-orbit of x under f, for this set is invariant under f. Define X_i $(i=0, 1, \cdots, k)$ to be the closure of the semi-orbit of $f^i(x)$ under f^k. It is readily verified that $f(X_i)=X_{i+1}$ $(i=0, 1, \cdots, k-1)$,

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[1] I wish to thank Professor G. A. Hedlund for his genial interest in the development of these results.

[2] Numbers in brackets refer to the bibliography at the end of the paper.

$f^k(X_i) \subset X_i$ $(i=0, 1, \cdots, k)$, and $X=\sum_{i=0}^{k-1} X_i$. We may suppose that $x \in \sum_{i=1}^{k-1} X_i$, for otherwise the conclusion follows. Let p, $1 \leqq p \leqq k-1$, be the smallest integer such that $x \in X_p$. Then, the semi-orbit of x under f^k is contained in X_p. Hence, $X_0 \subset X_p = f^p(X_0)$ and p is the smallest positive integer such that $X_0 \subset f^p(X_0)$. Since $f^k(X_0) \subset X_0$, there exists a smallest positive integer t such that $f^t(X_0) \subset X_0$. Now $f^t(X_0) \subset X_0 \subset f^p(X_0)$. Since $f^{t-p}(X_0) \subset X_0$ and $X_0 \subset f^{p-t}(X_0)$, p is neither less than nor greater than t. Hence, $p=t$ and $f^p(X_0)=X_0$. Write $k=pq+r$, $0 \leqq r < p$, where q and r are integers. Now

$$X_0 \supset f^k(X_0) = f^r(f^{pq}(X_0)) = f^r(X_0).$$

Thus, $r=0$ and $k=pq$. If $p=1$, then $X_0=X_1= \cdots =X_k$ whence $x \in X_k$ and the conclusion follows. We may suppose, therefore, that $p>1$. Now $p \leqq m$ and $q \leqq m$. By the induction assumption, x is recurrent under f^p and, applying the induction assumption to f^p, x is recurrent under $(f^p)^q=f^k$.

COROLLARY 1. *Every positive power of a pointwise recurrent homeomorphism on a topological space is itself pointwise recurrent.*

Theorem 1 can be used to provide a different proof of the following theorem, due to Birkhoff and Smith [2, p. 358, Theorem 3].

THEOREM. *If X is a compact metric space and if $f(X)=X$ is a homeomorphism, then for every nonzero integer n the central orbits under f^n are identical with the central orbits under f.*

This follows from Theorem 1 and the result, due to Birkhoff and Smith [2, p. 353, Theorem 2], that the sum of the central orbits under a homeomorphism h on a compact metric space is characterized as the closure of the set of points recurrent under both h and h^{-1}. Although their results on central orbits [2, pp. 350–355, 356–360] are stated for closed surfaces, their proofs are actually valid for compact metric spaces.

THEOREM 2. *If X is a compact connected metric space and if the recurrent points are dense in X, then every recurrent cut point x of X is periodic.*

PROOF. Express $X=A+B$, where A and B are nondegenerate continua such that $A \cdot B=x$. By a theorem due to Kelley [4, p. 194, Theorem 3.4] there exists an F-set (that is, either a simple link, or cut point, or end point) F of X such that $f(F)=F$. (For properties of simple links, see Whyburn [5, pp. 64–65].) Now F is contained in

either A or B, say A, and $A \cdot f^n(A) \neq \Lambda$ for every integer n. Since some point of $B-x$ is recurrent under f, there exists a positive integer n such that $B \cdot f^n(B) \neq \Lambda$. By Theorem 1, x is recurrent under f^n. Applying a lemma due to Whyburn [5, p. 247, Lemma 4.21], it follows that $f^n(x)=x$. The proof is completed.

Theorem 2 and its proof are partial generalizations of Whyburn [5, p. 248, Theorem 4.6], but the original conclusion—that the mapping is elementwise periodic on all simple links—is no longer valid without semi-local connectedness, even though the mapping be regularly almost periodic in the sense of Whyburn [5, p. 250]. Theorem 1, however, may be used as an aid in the proof of the cited theorem.

Remarks. 1. If the subset Y of X is minimal under f, then Y is minimal also under f^{-1}. 2. A nonvacuous subset Y of X is minimal if and only if the closure of the orbit of every point of Y is Y. 3. The collection of sets minimal under f is a decomposition of X if and only if the closure of the orbits under f is a decomposition of X; and, in either case, these two collections coincide. In other words, *f gives a minimal-set decomposition* if and only if *f gives an orbit-closure decomposition.*

THEOREM 3. *If X is minimal under f but not under f^k, where k is a nonzero integer, then there exists an integer n, $n>1$, such that n divides $|k|$ and f^n gives a finite minimal-set decomposition which contains exactly n elements.*

PROOF. By Remark 1, it is sufficient to prove the theorem when k is positive. There exists a point x of X such that the orbit of x under f^k is not dense in X, by Remark 2. Define X_i $(i=0, 1, \cdots, k-1)$ to be the closure of the orbit of $f^i(x)$ under f^k. Clearly, $f(X_i)=X_{i+1}$ $(i=0, 1, \cdots, k-2)$, $f(X_{k-1})=X_0$, and $f^k(X_i)=X_i$ $(i=0, 1, \cdots, k-1)$. Let p be the maximum positive integer such that there exist integers $i_1, i_2, \cdots, i_p$ with the properties that $0 \leqq i_1 < i_2 < \cdots < i_p \leqq k-1$ and $\prod_{j=1}^{p} X_{i_j} \neq \Lambda$. Choose integers $i_1, i_2, \cdots, i_p$ with these properties. Define $Y=\prod_{j=1}^{p} X_{i_j}$. Clearly, $f^k(Y)=Y$. Let n be the smallest positive integer such that $f^n(Y)=Y$. Define $Y_j=f^j(Y)$ $(j=0, 1, \cdots, n-1)$. The sets Y_j $(j=0, 1, \cdots, n-1)$ are closed and pairwise disjoint. Choose $y \in Y$. Then,

$$X = \overline{j \sum_{-\infty}^{+\infty} f^j(y)} = \overline{j \sum_{-\infty}^{+\infty} f^j(Y)} = \sum_{j=0}^{n-1} f^j(Y) = \sum_{j=0}^{n-1} Y_j.$$

Thus, $D \equiv [Y_j | j=0, 1, \cdots, n-1]$ is a decomposition of X.

We show $n>1$. Suppose $n=1$. Then, $X=Y_0=Y=X_{i_1}$, and thus

$X=f^{k-i_1}(X_{i_1})=f^k(X_0)=X_0$ whence the orbit of x under f^k is dense in X, contrary to the second statement of the proof.

We show that n divides k. Write $k=qn+r$, $0\leqq r<n$, where q and r are integers. Then,

$$Y = f^k(Y) = f^r(f^{qn}(Y)) = f^r(Y).$$

Hence, $r=0$.

In order to show that each element Y_j of D is minimal under f^n, it is sufficient to observe that for $y\in Y_j$, the orbit of y under f is dense in X, D is a decomposition of X whose elements are invariant under f^n, and the subset of the orbit of y under f which is contained in Y_j is actually the orbit of y under f^n.

COROLLARY 2. *If X is connected and minimal under f, then X is also minimal under f^n for every nonzero integer n.*

COROLLARY 3. *If X has only finitely many, say k, components and if X is minimal under f, then for every nonzero integer n the mapping f^n gives a finite minimal-set decomposition, the number of whose elements is the greatest common divisor of k and $|n|$.*

If $k=1$, Corollary 3 reduces to Corollary 2. If $k>1$, Corollary 3 may be proved by first of all considering the case when $k=k'$ and $n=n'>0$ are relatively prime and then extending the result to $k=ak'$ and $n=an'$, where a is any positive integer. Corollary 3 essentially combines Corollary 2 with a property of cyclic counting or, what is the same, a property of periodic orbits.

THEOREM 4. *If X is minimal under f, then for every nonzero integer n the mapping f^n gives a finite minimal-set decomposition of X into at most $|n|$ elements.*

PROOF. By Remark 1, it is sufficient to prove the theorem when n is positive. We make use of an induction. The theorem is true for $n=1$. Let m be any positive integer. Assume the theorem is true for $n\leqq m$. We now show the theorem is true for $n=m+1=k$.

If X is minimal under f^k, the conclusion follows. Suppose now that X is not minimal under f^k. By Theorem 3, there exist integers p and q such that $p>1$, $k=pq$, and f^p gives a finite minimal-set decomposition D of X into exactly p elements. Let Y be any element of D. Now apply the induction assumption to $f^p(Y)=Y$ and $n=q\leqq m$. Thus $(f^p)^q=f^k$ gives a finite minimal-set decomposition of Y into at most q elements. Hence, $(f^p)^q=f^k$ gives a finite minimal-set decomposition of X into at most $pq=k$ elements.

COROLLARY 4. *If the mapping f gives an orbit-closure decomposition of X, then for every integer n the mapping f^n also gives an orbit-closure decomposition of X.*

PROOF. By virtue of Remark 3, it is sufficient to apply Theorem 4 to the elements of the orbit-closure decomposition given by f.

LEMMA 1. *If X is a metric space and if $x \in X$ is almost periodic under f, then the closure Y of the orbit of x under f is minimal under f.*

PROOF. Suppose Y is not minimal. Then, there exists a nonvacuous closed invariant subset Z of Y such that $x \notin Z$. Choose $z \in Z$. Let 2ϵ be the distance from x to Z. There exists a positive integer N such that in every set of N consecutive positive integers appears an integer n so that $\rho(x, f^n(x)) < \epsilon$, where ρ is the metric in X. Choose $\delta > 0$ so small that $x' \in X$ with $\rho(z, x') < \delta$ implies $\rho(f^i(z), f^i(x')) < \epsilon$ $(i = 1, 2, \cdots, N)$. There exists an integer $p \geqq 0$ such that $\rho(z, f^p(x)) < \delta$. Also it is possible to find an integer q, $1 \leqq q \leqq N$, so that $\rho(x, f^{p+q}(x)) < \epsilon$. Furthermore, $\rho(f^q(z), f^{p+q}(x)) < \epsilon$. Hence, $\rho(x, f^q(z)) < 2\epsilon$ which is impossible because $f^q(z) \in Z$.

LEMMA 2. *If X is a compact metric space and if f gives an orbit-closure decomposition, then f is pointwise almost periodic.*

PROOF. Suppose that some point x of X is not almost periodic. Then there exist a neighborhood U of x and a sequence $m_1, m_2, \cdots$ of positive integers such that $U \cdot \sum_{j=1}^{+t} f^{m_i+j}(x) = \Lambda$ $(i = 1, 2, \cdots)$. We may suppose that the sequence $\{f^{m_i}(x)\}$ converges to some point, say y, of X. It is easy to show that the orbit of y is contained in $X - U$. Hence, the closure of the orbit of y is a proper subset of the closure of the orbit of x. This is impossible.

THEOREM 5. *If X is a metric space, then in order that f give an orbit-closure decomposition it is sufficient that f be pointwise almost periodic; and in case X is compact, this condition is also necessary.*

The proof follows easily from Lemmas 1 and 2 and Remark 3. Theorem 5 and Lemmas 1 and 2 are closely related to Hall and Kelley [3, p. 628, Theorem 4] and to Birkhoff [1, p. 199].

THEOREM 6. *Every power (including negative powers) of a pointwise almost periodic homeomorphism on a compact metric space is itself pointwise almost periodic.*

The proof follows readily from Theorem 5 and Corollary 4.

THEOREM 7. *If X is a compact metric space and if $x \in X$ is almost*

periodic under f, then x is also almost periodic under f^n for every integer n.

The proof proceeds easily from Lemma 1, Remark 2, and Theorems 5 and 6.

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SOME PROPERTIES OF SUMMABILITY. II[1]

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1. Summability of bounded sequences. It follows from a well known result of H. Steinhaus[2] that no regular matrix method of summability can be effective for (that is, assign a finite limit to) every element in the space (m) of bounded sequences. The object of this note is to consider some questions suggested by this fact. The first of these may be formulated as follows. If A is a given regular matrix method let J_A denote the set of all A-summable bounded sequences. We then ask what are necessary and sufficient conditions on a subset E of (m) in order that there exist a regular A such that $E \subset J_A$? In Theorem 1 below it is shown that the separability of E is a sufficient condition. It seems unlikely that this condition is necessary although we have been unable to decide the question. It is clearly equivalent to the question of whether every J_A is separable.

THEOREM 1. *Let E be an arbitrary separable subset of (m). Then every regular matrix $A=(a_{nk})$ contains a (necessarily regular) row-submatrix $B=(a_{n_i k})$ such that $E \subset J_B$.*

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[1] This note is in the nature of an appendix to the paper cited in footnote 4.

[2] H. Steinhaus, *Some remarks on the generalizations of the notion of limit* (in Polish), Prace Matematyczno-Fizyczne vol. 22 (1921) pp. 121–134. See also I. Schur, *Über lineare Transformationen in der Theorie der unendlichen Reihen*, J. Reine Angew. Math. vol. 151 (1921) pp. 79–111.

PROOF. The proof depends on an application of the so-called diagonal process. Let D denote a dense denumerable subset of E of points $x_n \equiv \{s_k^n\}$ for $n=1, 2, 3, \cdots$, and let $A=(a_{mk})$ be an arbitrary regular matrix. By the Silverman-Toeplitz conditions the sequence $\{t_m^1 \equiv \sum_k a_{mk}s_k^1\}$ is bounded and therefore contains a convergent subsequence $\{t_{m_i}^1\}$, where we may suppose that $m_1=1$. Let $a_{m_i k} \equiv a_{ik}^1$ for all i and k, and select a convergent subsequence $\{t_{i_i}^2\}$ from the bounded sequence $\{t_i^2 \equiv \sum_k a_{ik}^1 s_k^2\}$ so that $i_1=1$ and $i_2=2$. Next let $a_{i_j k}^1 \equiv a_{jk}^2$ for all j and k, proceed to $\{s_k^3\}$, and so forth. We define in this way a sequence of regular matrices (a_{mk}), (a_{mk}^1), (a_{mk}^2), $\cdots$, (a_{mk}^n), $\cdots$, each a row-submatrix of the preceding. The "diagonal matrix" $(a_{mk}^m) \equiv B$ is evidently regular and by its construction it is clearly effective for each x_n in D. To show that B is also effective in $E-D$ we first observe that the operation $y=B(x)$, where $B(x) \equiv \{\sum_k a_{mk}^m s_k\}$ and $x \equiv \{s_k\}$, is linear on (m) to (m). Then if x_0 is any point of $E-D$ there exists a subsequence $\{x_{n_i}\}$ of D that converges to x_0. Setting $y_i \equiv B(x_{n_i})$ we have $y_0 \equiv \lim_i y_i = \lim_i B(x_{n_i}) = B(x_0)$. Since y_i belongs to (c), a closed subset of (m), it follows that y_0 belongs to (c). This completes the proof.

2. **A property of reversible methods.** Let us first recall the following definitions. The method A corresponding to the matrix (a_{mk}) is said to be *reversible*[3] if the system of equations

$$t_m = \sum_{k=1}^{\infty} a_{mk}s_k \qquad (m = 1, 2, 3, \cdots) \tag{2.1}$$

has a unique solution $\{s_k\}$ corresponding to each convergent sequence $\{t_m\}$. The method A is said to be of[3] *type* M if the conditions

$$\sum_{m=1}^{\infty} |u_m| < \infty, \qquad \sum_{m=1}^{\infty} u_m a_{mk} = 0 \qquad (k = 1, 2, 3, \cdots)$$

always imply $u_m=0$ $(m=1, 2, 3, \cdots)$. Finally, every method A which is simultaneously regular, reversible, and of type M, is said to be *perfect*[3].

We now ask if every regular method of summability which is stronger than convergence is necessarily effective for some *unbounded* sequences. For the class of reversible matrix methods an affirmative answer is supplied by the following theorems. These theorems bring to light the fact that while perfect methods of summability may be equivalent to convergence, reversible regular methods not of type M are always stronger than convergence.

[3] S. Banach, *Théorie des opérations linéaires*, Warsaw, 1932, p. 90.

THEOREM 2. *Every perfect method of summability which is stronger than convergence is effective for some unbounded sequences.*

PROOF. Let $A=(a_{nk})$ be an arbitrary perfect matrix, stronger than convergence, and let F_A denote the set of all A-summable sequences. Then the transformation (2.1) of F_A into (c) has an inverse of the form[4]

$$s_k = \sum_{m=0}^{\infty} A_{km} t_m \qquad (k = 1, 2, 3, \cdots), \tag{2.2}$$

where $t_0 \equiv \lim_m t_m$ and $\sum_{m=0}^{\infty} |A_{km}| < \infty$ $(k=1, 2, 3, \cdots)$. If we let $x \equiv \{s_k\}$ and $y \equiv \{t_m\}$ we may express (2.1) as $y=A(x)$ and (2.2) as $x=A^{-1}(y)$.

Since A is of type M it follows from a lemma of Banach[5] that as x ranges over (c) the set G_A of all points $y=A(x)$ is dense in (c).

To give an indirect proof we assume now that $F_A \subset (m)$, that is, that every A-summable sequence is bounded. As a consequence it follows easily that the operation $x=A^{-1}(y)$ is then linear on (c) to F_A. Let x_0 be an arbitrary point of F_A and set $y_0=A(x_0)$. Since y_0 is in (c) there exists, by the foregoing remark, a sequence $\{y_n\}$ in G_A, where $y_n=A(x_n)$ for certain x_n in (c), such that $y_n \to y_0$. But then $x_n=A^{-1}(y_n) \to A^{-1}(y_0)=x_0$, and since (c) is complete we see that x_0 belongs to (c). Thus A is equivalent to convergence, and this contradiction completes the proof.

THEOREM 3. *Every reversible regular method A which is not of type M is effective for some unbounded sequences.*

PROOF. As x ranges over (m) we denote by H_A the set of all points $y=A(x)$. By an easy extension of the previously mentioned lemma of Banach, it has been shown[6] that A is of type M if and only if the points $z_n \equiv \{\delta_k^n\}$ for $n=1, 2, 3, \cdots$ (where δ_k^n is the Kronecker symbol) are points of $\overline{H}_A$, the closure of H_A. Hence if A is not of type M at least one of the points z_n, say z_r, is not in $\overline{H}_A$. Since A is reversible and z_r belongs to (c), the equation $A(x)=z_r$ has a unique solution $x=x_r$. It is clear that x_r can not be a point of (m) and this fact establishes the theorem.

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[4] See J. D. Hill, *Some properties of summability*, Duke Math. J. vol. 9 (1942) pp. 373–381; in particular, p. 376.

[5] See footnote 3, p. 93, Lemma 2.

[6] See J. D. Hill, *On perfect methods of summability*, Duke Math. J. vol. 3 (1937) pp. 702–714; in particular p. 704, Theorem 4.

An alternative proof of this theorem may be given as follows. Under the assumed hypotheses on A we conclude from known theorems that[7] there exists a regular method B, not weaker than A, with which A is inconsistent; and that[8] A and B are consistent for bounded sequences. Hence the inconsistency must occur for unbounded sequences.

3. **Methods effective for no bounded divergent sequences.** As our final question we ask if there exist regular methods, stronger than convergence, which are effective for *no* bounded divergent sequences. An example will suffice to show that the answer is affirmative.

Example. We define matrices $B=(b_{mk})$ and $C=(c_{mk})$ as follows: $b_{11}\equiv b_{k+1,k}\equiv 1$ $(k=1, 2, 3, \cdots)$, and $b_{mk}\equiv 0$ otherwise; $c_{mm}\equiv 1/m$ $(m=1, 2, 3, \cdots)$, and $c_{mk}\equiv 0$ otherwise. The matrix B is a regular matrix that leaves invariant the limit points of each bounded sequence. The matrix C, on the other hand, assigns the limit 0 to every bounded sequence. Consequently, the sum matrix $A\equiv B+C$ leaves invariant the limit points of each bounded sequence, and is therefore effective for no bounded divergent sequence. One easily verifies, however, that A assigns the limit 0 to the unbounded sequence $\{(-1)^{k-1}k!\}$.

The matrix A of this example is a simple instance of a Raff matrix,[9] namely, a matrix that leaves invariant the limit points of each bounded sequence. Every Raff matrix which is stronger than convergence will therefore possess the property in question, for example, every reversible Raff matrix which is not of type M (see Theorem 3 above).

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[7] See footnote 4, p. 380, Theorem 5.

[8] See footnote 3, p. 95, Théorème 11.

[9] Hermann Raff, *Zur Theorie der lineare Transformationen*, Math. Zeit. vol. 37 (1933) pp. 572–577.

THE TRANSFORMATION OF ČECH

V. G. GROVE

1. **Introduction.** The purpose of this paper is to give a simple construction of the general transformation of Čech [1, p. 192].[1]

Let the differential equations of a surface S be written in the Fubini canonical form [2, p. 123]

$$\begin{aligned} x_{uu} &= \theta_u x_u + \beta x_v + px, \\ x_{vv} &= \gamma x_u + \theta_v x_v + qx, \end{aligned} \qquad \theta = \log(\beta\gamma). \tag{1}$$

Let the differential equation defining a conjugate net N on S be written in the form

$$dv^2 - \lambda^2 du^2 = 0. \tag{2}$$

The ray and the associate ray intersect in *the canonical point* [3, p. 7] of N. The line joining the point x to the canonical point intersects the reciprocal of the Green-Fubini projective normal in a point whose coordinates are

$$(\beta/\lambda^2)x_u - \gamma\lambda^2 x_v. \tag{3}$$

We shall call this point *the conjugal point* of N at x.

2. **Conjugal quadrics.** Let the coordinates X of a point X be written in the form

$$X = x_1 x + x_2 x_u + x_3 x_v + x_4 x_{uv}.$$

Then with properly selected unit point, (x_1, x_2, x_3, x_4) are the coordinates of X referred to the tetrahedron (x, x_u, x_v, x_{uv}). The equation of the three-parameter family of quadrics each of which has second order contact [2, p. 142] with S at x is

$$x_2 x_3 + x_4(-x_1 + k_2 x_2 + k_3 x_3 + k_4 x_4) = 0. \tag{4}$$

The equation of any plane through the conjugal point (3) is

$$x_1 - k(\gamma\lambda^2 x_2 + (\beta/\lambda^2)x_3) - 2lx_4 = 0. \tag{5}$$

We shall speak of this plane as *the conjugal plane* of N at x.

If we impose the condition that the polar plane of the covariant point (0, 0, 0, 1) with respect to the quadric (4) be the conjugal plane

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[1] Numbers in brackets refer to the references cited at the end of the paper.

(5), we find that the equation of the quadric (4) assumes the form

$$(6) \qquad x_2x_3 + x_4[-x_1 + k(\gamma\lambda^2x_2 + (\beta/\lambda^2)x_3) + lx_4] = 0.$$

We shall call the quadric so determined *a conjugal quadric* of S at x with respect to the conjugate net N. There is a two-parameter family of such quadrics associated with a given conjugate net. It follows from (6) that *a given conjugal quadric may also serve as the conjugal quadric of the associate conjugate net of N only if it is a quadric of Darboux*. Such a quadric serves as the conjugal quadric of any conjugate net.

Davis has defined [3, p. 12] a pencil of quadrics by demanding that the tangents to the curves of intersection of a quadric of the family (4) with S be apolar to the tangents to the curves of a conjugate net. *The quadric of Davis for the conjugate net defined by* (2) *is the conjugal quadric* (6) *with* $k=1$. The quadric of Davis for the associate conjugate net of N is the conjugal quadric (6) with $k=-1$. We may call this quadric the *conjugate quadric* of Davis.

If one demands that a quadric of the family (4) have third order contact with each of the curves of N, the quadric must be a conjugal quadric with $k=-1/3$. The quadric with $k=1/3$ plays the same role with the associate conjugate net.

It will be recalled [2, pp. 185, 187] that the lines joining the points whose general coordinates are

$$x_u - bx, \qquad x_v - ax,$$

wherein

$$(7) \quad a = (\theta_v + \lambda_v/\lambda + m\beta/\lambda^2)/2, \qquad b = (\theta_u - \lambda_u/\lambda + m\gamma\lambda^2)/2,$$

are respectively the flex-ray, ray, associate ray, principal join and associate principal join according as

$$m = 0, -1, 1, 5/3, -5/3.$$

The lines joining x to the point whose coordinates are $x_{uv} - a'x_u - b'x_v$ with a', b' defined by

$$(8) \qquad a' = (\theta_v + \lambda_v/\lambda + n\beta/\lambda^2)/2, \quad b' = (\theta_v - \lambda_u/\lambda + n\gamma\lambda^2)/2$$

are respectively the axis, associate axis, cusp-axis, polar reciprocals of the principal join and associate principal join with respect to any quadric of Darboux according as $n=1, -1, 0, 5/3, -5/3$.

Imposing the condition that the lines determined by (7) be the polar lines of those defined by (8) with respect to the quadric (4) we find that quadric must be a conjugal quadric with

$$k = (n - m)/2. \tag{9}$$

It follows that the conjugal quadrics may be defined as quadrics having second order contact with S at x with respect to which canonical [3, p. 7] *lines of the first kind are reciprocal polars of canonical lines of the second kind.*

Using the various values of m and n for the well known canonical lines above we find interpretations for the conjugal quadrics for the particular values $k=0, \pm 1/3, \pm 1/2, \pm 5/6, \pm 1, \pm 4/3, \pm 5/3$.

In particular *the quadric of Davis is a quadric having second order contact with S at x and for which the axis of N is the polar of the ray of N.* Thus a new characterization of this quadric is obtained.

By means of the first characterization of the conjugal quadric with $k=-1/3$, we may give characterizations of some of the canonical lines associated with the net N. In particular *the principal join of the curves of N may be characterized as the polar line of the axis of the net with respect to the conjugal quadric $k=-1/3$; or it is the reciprocal of the associate ray of N with respect to the quadric of Davis.*

3. **The transformation of Čech.** In this section we shall show how the conjugal quadrics (6) may be considered as inducing the general transformation of Čech.

The coordinates of any point ρ on the tangent to the curve defined by $dv-\lambda du=0$ of the net N may be written in the form

$$\rho = x_u + \lambda x_v + \mu x.$$

The polar plane of ρ with respect to a conjugal quadric (6) has local coordinates given by the formulas

$$u_1 = 0, \qquad u_2 = \lambda^2, \qquad u_3 = \lambda, \qquad u_4 = -\lambda\mu + k(\beta + \gamma\lambda^2). \tag{10}$$

If the local coordinates of ρ be written in the form $(x_1, x_2, x_3, 0)$, equations (10) assume the form

$$\begin{aligned} &u_1 = 0, \qquad u_2 = x_2\overset{2}{x_3}, \qquad u_3 = \overset{2}{x_2}x_3, \\ &u_4 = -x_1x_2x_3 + k(\beta\overset{3}{x_2} + \gamma\overset{3}{x_3}) \end{aligned} \tag{11}$$

of general transformation of Čech. We shall say that *the conjugal quadrics* (6) *induce the general transformation of Čech.* In particular *the quadrics of Darboux induce the polarity of Lie, $k=0$,* and *the quadrics of Davis induce the correspondence of Segre, $k=1$.*

The associate quadric of Davis may be said to induce *the associate correspondence of Segre, $k=-1$. The conjugal quadric $k=-1/3$ induces the correspondence of Moutard.* We may call the transformation

(11) with $k=1/3$ *the associate correspondence of Moutard*. The quadric (6) with $k=1/3$ induces this latter correspondence. The other particular conjugal quadrics characterized in the previous section of course induce the corresponding transformations of Čech.

Lane has given [2, p. 203] certain properties of the general transformation of Čech. We call attention to one additional property. We may easily verify that the most general linear transformation which leaves the form of the equations of the family of conjugal quadrics (6) invariant is the transformation

$$\begin{aligned} \sigma x_1 &= \bar{x}_1 - b\bar{x}_2 - a\bar{x}_3 - c\bar{x}_4, \\ \sigma x_2 &= \bar{x}_2 - a\bar{x}_4, \qquad \sigma x_3 = \bar{x}_3 - b\bar{x}_4, \qquad \sigma x_4 = \bar{x}_4. \end{aligned} \tag{12}$$

The transformation (12) and the corresponding transformation in plane coordinates leave the form of each of the transformations of Čech (11) absolutely invariant. *It follows therefore from the form of* (12) *that the transformation of the form* (11) *obtained by using any R-harmonic line* [4, p. 584] *and its reciprocal in the definition of the conjugal quadrics is the transformation of Čech for the same value of k.*

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DISTANCE GEOMETRY NOTES

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Introduction. This paper is concerned with several disconnected developments in distance geometry. §1 deals with the congruent imbedding of metric spaces in euclidean or Hilbert spaces. By showing that *the validity of the pythagorean theorem insures the essentially euclidean character of the metric*, the basic role this theorem plays in euclidean geometry is seen to be fully justified. In §2 the circle and n-dimensional sphere are considered with respect to the property of covering euclidean subsets. The concluding section presents an algebraic-geometric proof of the quasi congruence order property of the E_n which, by making use of determinants, achieves a considerable abbreviation of the two proofs of this important result hitherto published. The desired purely algebraic proof has not yet been obtained.

1. Metric spaces and the theorem of Pythagoras. If a metric space contains a line L (that is, a set congruent with the euclidean straight line) and a point p not on L, then the line contains a point p_0 nearest p. The pythagorean theorem is valid in the space provided for each element x of L, $pp_0^2+p_0x^2=px^2$. It is clear that this property insures the uniqueness of the point p_0.

THEOREM 1.1. *A separable, complete, convex, externally convex metric space in which the theorem of Pythagoras is valid is congruent with a euclidean or Hilbert space.*

PROOF. From an early result of the writer it suffices to show that the space has the weak euclidean four-point property; that is, each quadruple of points containing a linear triple is congruently contained in the plane.[1] If p, q, r, s are four such points, L a line containing q, r, s, and p_0 the point of L nearest p, let q', r', s', p_0' be points of an E_1 congruent with the set q, r, s, p_0. In the plane E_2 formed by this E_1 and the line perpendicular to it at p_0', let p' denote a point on this perpendicular with $p'p_0'=pp_0$. Since the theorem of Pythagoras is valid in both the given space and in E_2, it is clear that the distances of p' from the points q', r', s' equal, respectively, the distances of p from q, r, s, and the theorem is proved.

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[1] See Amer. J. Math. vol. 57 (1935) pp. 51–61; in particular, p. 61.

It is worth remarking that the proof of the theorem, trivial when based upon the *weak* euclidean four-point property, is by no means immediate in terms of the ordinary four-point property which necessitates showing that *every* quadruple is congruently contained in E_3. It serves, therefore, as a good example of the usefulness of the weaker property.

If any condition leading to finite dimensionality is adjoined to those in the theorem (for example the absence of equilateral $(n+2)$-tuples or the compactness of bounded sets) the condition of separability may be suppressed (completeness also in the second alternative, of course) and congruence with a euclidean space results.

2. **Covering indices.** A given space is said to have congruence indices (n, k) with respect to a prescribed class of spaces provided any space of the class, containing more than $n+k$ points, is congruently contained in the given space whenever each of its n-tuples has this property. The indices (n, k) are "best" in case neither can be reduced.[2]

If the given space is a subset of a euclidean space and the spaces of the prescribed class are also euclidean subsets (though not necessarily of the same space of which the given space is a subset) the possession of congruence indices (n, k) means that any member of the class, containing more than $n+k$ points, can be covered by the given space whenever each n of its points is coverable by the space. This suggests the term "covering indices" as a specialization of the more general concept. Thus, for example, a circular disc has covering indices (3, 0) with respect to the class of subsets of the plane, since any plane set may be covered by such a disc whenever each three of its points are so coverable. Denoting by $C_{n,r}$ the surface of the n-dimensional sphere of radius r, with euclidean (chord) metric, we have:

LEMMA 2.1. *If each three of four distinct coplanar points are coverable by the circle $C_{1,r}$, but the four points are not so coverable, then they form an orthocentric quadruple.*

PROOF. Let p_1, p_2, p_3, p_4 denote the four points, and q_i the circumcenter of the triple obtained by omitting the ith point $(i=1, 2, 3, 4)$ (clearly no three of the points are on a line). It follows at once from the hypotheses that q_1, q_2, q_3 are reflections of q_4 in the sides p_2p_3, p_1p_3, p_1p_2, respectively, of the triangle $p_1p_2p_3$, and $p_4q_1=p_4q_2=p_4q_3=r$. Thus p_4 is the isogonal conjugate of the circumcenter q_4

[2] The concept of congruence indices was introduced in Bull. Amer. Math. Soc. vol. 47 (1941) pp. 435–443.

of triangle $p_1p_2p_3$ and hence is the orthocenter of the triangle.

If four points form an orthocentric quadruple, the four triples have circumcircles of the same radius. But the four points are surely not concyclic, for since p_4, for example, is the isogonal conjugate of the circumcenter q_4 of the triangle $p_1p_2p_3$, it follows that q_4 is the isogonal conjugate of p_4 which cannot then lie on the circumcircle of $p_1p_2p_3$ since no point on this circle has an isogonal conjugate with respect to $p_1p_2p_3$. *Hence a necessary and sufficient condition that four distinct coplanar points be not coverable by $C_{1,r}$, though each triple of the points be coverable, is that the four points form an orthocentric quadruple.*

THEOREM 2.1. *The $C_{1,r}$ has best covering indices* (3, 1) *with respect to subsets of the plane.*

PROOF. To prove the validity of indices (3, 1) it suffices to show that any planar quintuple is coverable by $C_{1,r}$ whenever each three of its points are. If, now, $p_1, \cdots, p_5$ be such a quintuple, an assumption that it is not coverable by $C_{1,r}$ implies that at least two of its quadruples be not coverable. According to the foregoing, each of these quadruples forms an orthocentric set, and since they have a triple in common, the fourth points in each quadruple coincide, for each is the orthocenter of the common triple.

That the indices (3, 1) are best is obvious since it is trivial to observe that the first index cannot be reduced, while the example of an orthocentric quadruple shows that the second index cannot be replaced by zero.

LEMMA 2.2. *If four noncoplanar points have each triple coverable by $C_{1,r}$, then they are the vertices of an isosceles tetrahedron.*

PROOF. Let O denote the center of the sphere circumscribing the four points and R its radius. The feet of the perpendiculars from O to the faces of the tetrahedron are the circumcenters of these faces and so the distances of O from each face is $(R^2-r^2)^{1/2}$. Thus O is also the center of the inscribed sphere and the assertion of the lemma follows.

LEMMA 2.3. *The $C_{1,r}$ has best covering indices* (3, 1) *with respect to subsets of E_3.*

PROOF. Let S be any subset of E_3 containing more than four points, each three of which are coverable by $C_{1,r}$, and let $p_1, \cdots, p_5$ be any five points of S. If these five points are not coverable by $C_{1,r}$ then, by Theorem 2.1, the five points are not coplanar and hence they contain at most one planar quadruple. Examining the (at least) four nonplanar quadruples in the light of Lemma 2.2, one sees that all the

ten distances determined by the five points are equal. Since the E_3 contains no equilateral five-point, this is impossible and the lemma follows. (Since the indices (3, 1) are best with respect to subsets of E_2 they are a fortiori best for subsets of E_3.)

It is noteworthy that $C_{1,r}$ has the same set of best indices with respect to subsets of E_2 as it has with respect to subsets of E_3.

THEOREM 2.2. *The $C_{1,r}$ has best covering indices $(3, n-2)$ with respect to subsets of E_n, $n>2$.*

PROOF. This has been proved for $n=3$. We make the inductive assumption of the theorem's validity in E_n, $n>2$. Let $p_1, p_2, \cdots, p_{n+3}$ be any $(n+3)$-tuple of a subset S of E_{n+1} containing more than $n+2$ points, with each triple coverable by $C_{1,r}$. If this $(n+3)$-tuple be supposed not coverable by $C_{1,r}$ it follows (by the inductive hypothesis) that the $(n+3)$-tuple is not contained in E_n.

Case 1. *No $(n+2)$-tuple of $p_1, p_2, \cdots, p_{n+3}$ is in E_n.*

Then no quadruple of these points is in E_2 and hence (Lemma 2.2) for each quadruple p_i, p_j, p_k, p_l, "opposite" distances are equal. Then the $n+3$ points form an equilateral set, which is impossible since they are contained in E_{n+1}.

Case 2. *At least one $(n+2)$-tuple of $p_1, p_2, \cdots, p_{n+3}$ is contained in E_n.*

Since the $(n+3)$-tuple is not contained in E_n, it follows that in this case exactly one $(n+2)$-tuple is contained in E_n, say $p_2, \cdots, p_{n+3}$. By the inductive hypothesis these $n+2$ points are coverable by $C_{1,r}$, and hence they lie in a plane. But then the $n+3$ points lie in E_3, which (since $n>2$) contradicts the conclusion reached above that $p_1, p_2, \cdots, p_{n+3}$ are not contained in E_n.

Hence the $C_{1,r}$ has covering indices $(3, n-2)$ with respect to subsets of E_n, $n>2$. The presence in E_n of equilateral $(n+1)$-tuples with each three points coverable by $C_{1,r}$ shows that the indices $(3, n-2)$ are best.

THEOREM 2.3. *The $C_{n,r}$ has covering indices $(n+2, 1)$ with respect to subsets of E_{n+1}.*

PROOF. Let S be a subset of E_{n+1} containing more than $n+3$ points and consider the semimetric space $S'+(a')$, with S' congruent with S and $a'p'=r$ for each element p' of S'.

It is seen at once that each set of $n+3$ points of this semimetric space is congruently contained in E_{n+1}, for if the $n+3$ points are in S' this follows from the congruence of S' with S, while if $p_1', p_2', \cdots, p_{n+2}'$ are any $n+2$ points of S', the $(n+3)$-tuple

p_1', p_2', $\cdots$, p_{n+2}', a' is congruent with $p_1, p_2, \cdots, p_{n+2}, a$, where the points $p_1, p_2, \cdots, p_{n+2}$ are selected from S and the point a is the center of the $C_{n,r}$ passing through them.

Since $S'+(a')$ contains more than $n+4$ points, and E_{n+1} has, by a fundamental theorem (proof of which is given in §3), congruence indices $(n+3, 1)$ with respect to the class of all semimetric spaces, it follows that this semimetric space is congruently contained in E_{n+1}, and as each point of S' has distance r from a', it is coverable by $C_{n,r}$ and the theorem is proved.

It seems quite likely that the indices $(n+2, 1)$ are best, though this has not as yet been completely established except for $n=1, 2$.

It is expected that these covering properties will lead to new characterizations of the figures concerned. It has already been shown, for example, that the circular disc is characterized among all connected, simply connected domains of the plane by the property of possessing covering indices (3, 0) with respect to plane sets. In an analogous manner, the circular rim $C_{1,r}$ might be characterized among all simple closed curves of the plane by possessing best covering indices (3, 1) with respect to planar subsets (or, perhaps, among all Jordan curves of E_n by having as best indices $(3, n-2)$).[3]

3. Congruence indices of the E_n with respect to semimetric spaces. It was shown by Menger that any semimetric space of more than $n+3$ points is congruently contained in E_n whenever each $n+2$ of its points has this property. Thus E_n has congruence indices $(n+2, 1)$ with respect to the class of semimetric spaces or, in the older terminology, quasi congruence order $n+2$. This theorem is fundamental in the metric study of euclidean space.

The literature contains only two proofs of this important result, both of which are quite lengthy and necessitate considerable indoctrination of the reader in the preceding theory.[4] This is due, in part, to the fact that the argument is entirely geometrical and demonstrates more than is actually needed in order to arrive at the desired result. Since the theorem is easily formulated in the language of determinant or quadratic form theory, attempts have been made to furnish a completely algebraic proof of the theorem. The develop-

[3] In this connection it would be of interest to prove or disprove the writer's conjecture that corresponding to each plane simple closed curve there exists a circle with each three of its points coverable by the simple closed curve. In the event that the curve is a triangle, the inscribed circle serves as the circle of the conjecture.

[4] The proofs are by Karl Menger, Math. Ann. vol. 100 (1928) pp. 120–130 and Amer. J. Math. vol. 53 (1931) pp. 730–737.

ments presented here do not accomplish this but they do employ determinant theory to considerably shorten the proofs heretofore obtained, and they hew close to the line by proving only those facts which are needed to establish the result. Thus the end is achieved more speedily though the route travelled is not essentially different.[5]

Calling a set pseudo-E_n provided it is not congruently contained in E_n though each of its $(n+2)$-tuples is, we show first the following lemma.

LEMMA 3.1. *A pseudo-E_n $(n+4)$-tuple P contains at least $n+2$ pseudo-E_n $(n+3)$-tuples.*

PROOF. Suppose this is not the case. Then P contains at least three $(n+3)$-tuples which are congruently contained in E_n. Now since any three $(n+3)$-tuples of P have $n+1$ points in common, we may assume the labelling so that $p_1, p_2, \cdots, p_{n+1}, p_{n+2}, p_{n+3}$; $p_1, p_2, \cdots, p_{n+1}, p_{n+2}, p_{n+4}$; $p_1, p_2, \cdots, p_{n+1}, p_{n+3}, p_{n+4}$ are euclidean $(n+3)$-tuples. Then the determinant $D(p_1, p_2, \cdots, p_{n+3}, p_{n+4})$ of the set P has rank less than or equal to $n+2$, since the principal minors $D(p_1, p_2, \cdots, p_{n+1}, p_{n+i})$ vanish for $i=2, 3, 4$ (every $n+2$ points of P being euclidean) and

$$D(p_1, \cdots, p_{n+1}, p_{n+2}, p_{n+3}) = D(p_1, \cdots, p_{n+1}, p_{n+2}, p_{n+4}) = D(p_1, \cdots, p_{n+1}, p_{n+3}, p_{n+4}) = 0$$

(since the three $(n+3)$-tuples are euclidean).

It follows that each $(n+3)$-tuple of P has a vanishing determinant D and hence is congruently contained in E_n. But this implies that P is euclidean, contrary to the hypothesis.

LEMMA 3.2. *Let P be a pseudo-E_n $(n+3)$-tuple. Then each $n+2$ points of P contains an independent $(n+1)$-tuple.*

PROOF. If this is not the case we may assume the labelling so that $p_1, p_2, \cdots, p_{n+2}$ are $n+2$ points of P with each of its $(n+1)$-tuples dependent. Then $D(p_1, p_2, \cdots, p_{n+2})$ has rank not exceeding $n+1$, for $D(p_1, p_2, \cdots, p_{n+2})$ vanishes and $D(p_1, p_2, \cdots, p_n, p_{n+1}) = D(p_1, p_2, \cdots, p_n, p_{n+2}) = D(p_1, p_2, \cdots, p_n, p_{n+1}, p_{n+2}) = 0$. Thus all first minors of $D(p_1, p_2, \cdots, p_{n+2})$ vanish, and hence $D(p_1, p_2, \cdots, p_{n+2})$ is zero. This is impossible since P is a pseudo-E_n $(n+3)$-tuple.

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[5] We are referring here to the second of the two references in the preceding footnote.

LEMMA 3.3. *Let P and Q be two pseudo-E_n $(n+3)$-tuples with $n+2$ points of P congruent with $n+2$ points of Q. Then P and Q are congruent.*

PROOF. By the preceding lemma the $n+2$ points of P which are congruent to $n+2$ points of Q contain an independent $(n+1)$-tuple. Selecting the labelling so that the $n+2$ points $p_1, p_2, \cdots, p_{n+1}, p_{n+2}$ of P are congruent with the $n+2$ points $q_1, q_2, \cdots, q_{n+1}, q_{n+2}$ of Q, it follows easily that the distances of p_{n+3} from the points of the first set are the same functions of the mutual distances of $p_1, p_2, \cdots, p_{n+2}$ as the distances of q_{n+3} from the points of the congruent set $q_1, q_2, \cdots, q_{n+2}$, and hence the two $(n+3)$-tuples are congruent.

THEOREM 3.1. *If a semimetric space S of more than $n+3$ points has each of its $(n+2)$-tuples congruently contained in E_n, then S is congruently contained in E_n.*

PROOF. Suppose S is not congruently contained in E_n. Then S has at least one $(n+3)$-tuple with this property, and any $(n+4)$-tuple containing this $(n+3)$-tuple is a pseudo-E_n set. Let $P=(p_1, p_2, \cdots, p_{n+4})$ be such an $(n+4)$-tuple. By Lemma 3.1 at most two of the $(n+3)$-tuples of P can be euclidean, say $p_1, p_2, \cdots, p_{n+2}, p_{n+3}$ and $p_1, p_2, \cdots, p_{n+3}, p_{n+4}$.

To show that $p_1, p_2, \cdots, p_{n+2}$ forms an equilateral set (which furnishes the desired contradiction to the assumption that S is not congruently contained in E_n) let p_ip_j, p_rp_s be any pair of its nonzero distances. Then the indices i, j, r, s are selected from the numbers $1, 2, \cdots, n+2$, and so the two $(n+3)$-tuples $p_1, p_2, \cdots, p_{i-1}, p_{i+1}, \cdots, p_{r-1}, p_{r+1}, \cdots, p_{n+3}, p_{n+4}, p_i$; $p_1, p_2, \cdots, p_{i-1}, p_{i+1}, \cdots, p_{r-1}, p_{r+1}, \cdots, p_{n+2}, p_{n+3}, p_{n+4}, p_r$ are pseudo-E_n. From Lemma 3.3 we have $p_ip_j=p_rp_j$.

But from the two pseudo-E_n $(n+3)$-tuples $p_1, p_2, \cdots, p_{j-1}, p_{j+1}, \cdots, p_{s-1}, p_{s+1}, \cdots, p_{n+2}, p_{n+3}, p_{n+4}, p_j$; $p_1, p_2, \cdots, p_{j-1}, p_{j+1}, \cdots, p_{s-1}, p_{s+1}, \cdots, p_{n+2}, p_{n+3}, p_{n+4}, p_s$ we have $p_jp_r=p_sp_r$. Hence $p_ip_j=p_rp_s$ and the $n+2$ points $p_1, p_2, \cdots, p_{n+2}$ form an equilateral $(n+2)$-tuple. They are not, then, congruently contained in E_n, contrary to the hypothesis, and the theorem is proved.

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TOPOLOGICAL ANALOG OF THE WEIERSTRASS DOUBLE SERIES THEOREM

G. T. WHYBURN

Since any light interior transformation of a sphere or a Riemann surface into a sphere is topologically equivalent to an analytic transformation and any non-constant analytic transformation is light and interior,[1] it might be expected in view of the Weierstrass Double Series Theorem that the limit of a uniformly convergent sequence of light interior mappings of a sphere into a sphere would be light and interior. However, this is readily seen not to be the case. For if we let r denote $|z|$ and for each $n>0$ we map the complex sphere onto itself by the function $f_n(z)$ defined by

$$w = z \qquad \text{for } r \geqq 2,$$
$$w = z/n \qquad \text{for } r \leqq 1,$$
$$w = (r-1)z + (2-r)(z/n) \qquad \text{for } 1 < r < 2,$$

each mapping is topological, whereas the sequence $f_n(z)$ converges uniformly to the mapping $f(z)$:

$$w = z \qquad \text{for } r \geqq 2,$$
$$w = 0 \qquad \text{for } r \leqq 1,$$
$$w = (r-1)z \qquad \text{for } 1 < r < 2.$$

The latter mapping is neither light nor interior. (Of course, analytic mappings topologically equivalent to the transformations $f_n(z)$ can be chosen so as to exhibit a wide range of behaviors, since they need only be topological mappings.)

It will be noted in this example, however, that if we factor the limit mapping $w=f(z)$ into the form $f_2f_1(z)$ where f_1 is monotone and f_2 is light,[2] *the transformation f_2 is light and interior.* This suggests the possibility, which will indeed be realized in the much more general situation where the mappings operate on an arbitrary locally connected continuum, that the limit of a uniformly convergent sequence of light interior mappings always factors into a monotone transformation followed by a light interior one. In fact, it will be shown that if the

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[1] See S. Stoïlow, *Leçons sur les principes topologiques de la théorie des fonctions analytiques*, Paris, 1938, Gauthier-Villars.

[2] See, for example, the author's *Analytic topology*, Amer. Math. Soc. Colloquium Publications, New York, 1942, chap. 8. Note references therein also to S. Eilenberg.

individual mappings in the sequence admit of such factorization, so also does their limit.

Now it was shown by Wallace[3] that a continuous transformation $f(A)=B$ on a locally connected continuum A factors into this form if and only if it is quasi-monotone, that is, provided the inverse of each continuum K in B having a non-empty interior relative to B consists of a finite number of components each of which maps onto all of K under f or, equivalently, provided that for each connected open set R in B each component of $f^{-1}(R)$ maps onto R under f. Thus our main conclusion asserts in these terms that the class of quasi-monotone mappings of a locally connected continuum A onto another one B is closed in the space B^A of all mappings of A into B. The Weierstrass Double Series Theorem asserts, of course, that the class of analytic transformations is closed. In view of the equivalence theorem of Stoïlow[1] our result is analogous, except that the "constants" (more precisely, "the continua on which the mappings are constant") must first be factored out.

THEOREM. *The limit mapping $f(A)=B$ of a uniformly convergent sequence of quasi-monotone mappings $f_n(A)=B$ on a locally connected continuum A is itself quasi-monotone.*

In view of the fact mentioned above that a continuous mapping $g(A)=B$ on a locally connected continuum A is quasi-monotone if and only if each component of the inverse of an arbitrary region (that is, connected open subset) R in B maps onto R under g, our theorem is a direct consequence of the following more general one.

THEOREM. *Let the sequence of continuous mappings $f_n(A)=B$ on the locally connected continuum[4] A converge uniformly to the mapping $f(A)=B$. If for each $\epsilon>0$ there exists an integer N such that for any $n>N$ it is true that if R is any region in B and Q is any component of $f_n^{-1}(R)$, then*

$$f_n(Q)\cdot\{R-R\cdot V_\epsilon[F(R)]\}\neq 0$$

implies

$$f_n(Q)\supset R-R\cdot V_\epsilon[F(R)],$$

then f is quasi-monotone.

[3] A. D. Wallace, *Quasi-monotone transformations*, Duke Math. J. vol. 7 (1940) pp. 136–145.

[4] It is understood that a continuum is compact, metric and connected. For any set X and any $\epsilon>0$, $V_\epsilon(X)$ denotes the set of all points x whose distance $\rho(x, X)$ from X is less than ϵ. Also if G is any open set, $F(G)$ denotes its boundary, that is, $\bar{G}-G=F(G)$.

PROOF. To prove f quasi-monotone it suffices, in view of the factorization process, to show that for any $y \in B$ and any neighborhood V of a component X of $f^{-1}(y)$, $f(V)$ contains a neighborhood of y. By local connectedness, there exists a region U in A such that

$$V \supset \overline{U} \supset U \supset X.$$

Further, since X is a component of $f^{-1}(y)$, we may choose U also so that if $F = F(U)$, $f(F) \cdot y = 0$. Let R be a region in B containing y and such that $\overline{R} \cdot f(F) = 0$. Let

$$\text{(i)} \qquad 3\epsilon = \rho[R, f(F)], \qquad F_n = f_n(F).$$

By uniform convergence there exists an integer N such that for any $n > N$,

$$\text{(ii)} \qquad F_n \subset V_\epsilon[f(F)], \qquad f_n(X) \subset R.$$

Now for any $n > N$, let R_n be the component of $B - F_n$ containing R and let Q_n be the component of $f_n^{-1}(R_n)$ containing X. By (ii) we have

$$f_n(Q_n) \cdot R \neq 0.$$

Whence by (i)

$$f_n(Q_n) \cdot \{R_n - R_n \cdot V_\epsilon[F_n]\} \neq 0;$$

and since

$$F_n \supset F(R_n),$$

this gives

$$f_n(Q_n) \cdot \{R_n - R_n \cdot V_\epsilon[F(R_n)]\} \neq 0.$$

By hypothesis this implies that for each $n > N$,

$$f_n(Q_n) \supset R_n - R_n \cdot V_\epsilon[F(R_n)] \supset R.$$

Thus, since $Q_n \subset U$,

$$f_n(\overline{U}) \supset f_n(Q_n) \supset R,$$

for every $n > N$.

Now by uniform convergence, $f_n(\overline{U})$ converges to $f(\overline{U})$. Thus we have

$$f(\overline{U}) = \lim f_n(\overline{U}) \supset R,$$

so that

$$f(V) \supset f(\overline{U}) \supset R,$$

as was to be proven.

COROLLARY. *If the sequence $f_n(A) = B$ of light interior mappings on a locally connected continuum A converges uniformly to the mapping $f(A) = B$, then f admits of unique factorization into the form $f = f_2 f_1$ where f_1 is monotone and f_2 is light and interior.*

The case in which A is a topological sphere is of special interest, not only because of its connection with analytic functions but also because in this case the action of the separate factor mappings can be more completely analyzed. For the monotone factor $f_1(A) = A'$ always yields[5] a cactoid A' and then the light interior factor $f_2(A') = B$ operating on the cactoid A' lends itself readily[6] to analysis.

In conclusion it may be noted that our main conclusion obtained here for monotone mappings is known[7] to hold also for monotone mappings, that is, the limit of a uniformly convergent sequence of monotone mappings of a locally connected continuum A onto another one B is monotone.

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[5] See R. L. Moore, *Concerning upper semi-continuous collections*, Monatshefte für Mathematik und Physik vol. 36 (1929) pp. 81–88.

[6] See footnote 2, chap. 10.

[7] Footnote 2, p. 174.

THE "FUNDAMENTAL THEOREM OF ALGEBRA" FOR QUATERNIONS

SAMUEL EILENBERG AND IVAN NIVEN

We are concerned with polynomials of degree n of the type

$$f(x) = a_0xa_1x \cdots xa_n + \phi(x),$$

where $x, a_0, a_1, \cdots, a_n$ are real quaternions ($a_i \neq 0$ for $i = 0, 1, \cdots, n$) and $\phi(x)$ is a sum of a finite number of similar monomials $b_0xb_1x \cdots xb_k$, where $k < n$.

THEOREM 1. *The equation $f(x) = 0$ has at least one solution.*[1]

The 4-dimensional euclidean space R_4 of all quaternions will be made compact by adding the point ∞ to form a 4-dimensional sphere S_4. Setting $f(\infty) = \infty$ we get a mapping

$$f: S_4 \to S_4.$$

The continuity of f at ∞ follows from the fact that $|f(x)|$ increases without limit as $|x|$ increases without limit, a fact which is obvious from the definition of f. It should be noted that this argument is not valid for polynomials of degree n with more than one term of degree n.

Theorem 1 is then a consequence of the following theorem which asserts that "essentially" the equation $f(x) = 0$ has exactly n solutions.

THEOREM 2. *The mapping $f: S_4 \to S_4$ has the degree n (in the sense of Brouwer).*[2]

We define a mapping $g: S_4 \to S_4$ as follows:

$$g(x) = x^n \quad \text{for} \quad x \in R_4, \qquad g(\infty) = \infty.$$

Theorem 2 is a consequence of the following two lemmas.

LEMMA 1. *The mappings f and g of S_4 into S_4 are homotopic.*

LEMMA 2. *The mapping g has degree n.*

PROOF OF LEMMA 1. Define for $0 \leqq t \leqq 1$

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[1] This result has been obtained for the special case in which all terms of $f(x)$ have the form ax^k; cf. Ivan Niven, *Equations in quaternions*, Amer. Math. Monthly vol. 48 (1941) pp. 654–661.

[2] Cf. Alexandroff and Hopf, *Topologie*, Berlin, 1935, chap. 12, for the theory of the degree of a mapping.

$$f_t(x) = a_0 x a_1 x \cdots x a_n + (1-t)\phi(x) \quad \text{for} \quad x \in R_4,$$
$$f_t(\infty) = \infty.$$

Clearly $f_t(x)$ is continuous in both x and t, and consequently f is homotopic to the mapping

$$f_1(x) = a_0 x a_1 x \cdots x a_n \quad \text{for} \quad x \in R_4,$$
$$f_1(\infty) = \infty.$$

Now for each $i=0, 1, \cdots, n$ choose a continuous path $a_{i,t}$ in $R_4-(0)$, where $1 \leqq t \leqq 2$, with end points $a_{i,1}=a_i$ and $a_{i,2}=1$. Define for $1 \leqq t \leqq 2$

$$f_t(x) = a_{0,t} x a_{1,t} x \cdots x a_{n,t} \quad \text{for} \quad x \in R_4,\ f_t(\infty) = \infty.$$

Again $f_t(x)$ is continuous in x and t and since $f_2(t)=g(t)$ the conclusion follows.

PROOF OF LEMMA 2. It is known[3] that the equation

$$g(x) = i, \quad \text{that is,} \quad x^n = i, \tag{1}$$

has exactly n roots in quaternions, and hence these must be simply the nth roots of i in complex numbers. We shall compute the Jacobian of the mapping function g for the roots of (1), and show that it is positive at each of these roots. This imples our conclusion.[4]

Any quaternion $x=x_1+x_2i+x_3j+x_4ij$ can be written in the form $x=k(\cos\theta+\tau\sin\theta)$, where

$$k = (x_1^2 + x_2^2 + x_3^2 + x_4^2)^{1/2},$$
$$\sin\theta = \pm (x_2^2 + x_3^2 + x_4^2)^{1/2}/k, \qquad \cos\theta = x_1/k,$$
$$\tau = \pm (x_2 i + x_3 j + x_4 ij)/(x_2^2 + x_3^2 + x_4^2)^{1/2}.$$

The last equation is invalid if $x_2^2+x_3^2+x_4^2=0$; in this case let τ be any unit vector. The sign used in the last two equations is arbitrary; however, in the special case to be considered in which x represents any nth root of i, it will be convenient to let this sign coincide with that of x_2. Now we have

$$x^n = k^n(\cos n\theta + \tau \sin n\theta),$$

and writing $x^n=f_1+f_2i+f_3j+f_4ij$ we obtain

$$f_1 = f_1(x_1, x_2, x_3, x_4) = k^n \cos n\theta,$$

[3] Cf. Ivan Niven, *The roots of a quaternion*, Amer. Math. Monthly vol. 49 (1942) pp. 386–388, or Louis Brand, a paper with the same title, same journal and volume, pp. 519, 520.

[4] Cf. Alexandroff and Hopf, p. 477.

$f_r = f_r(x_1, x_2, x_3, x_4) = k^n x_r \sin n\theta / \pm (x_2^2 + x_3^2 + x_4^2)^{1/2}$ $(r = 2, 3, 4)$.

Computing $\partial f_r/\partial x_s$ $(r, s = 1, 2, 3, 4)$, and using the following relations which hold for all the roots of (1), $k = 1$, $\sin n\theta = 1$, $\cos n\theta = 0$, $\sin \theta = x_2$, $\cos \theta = x_1$, $x_3 = x_4 = 0$, $\tau = i$, we find that the Jacobian of the system is

$$\begin{vmatrix} nx_2 & -nx_1 & 0 & 0 \\ nx_1 & nx_2 & 0 & 0 \\ 0 & 0 & 1/x_2 & 0 \\ 0 & 0 & 0 & 1/x_2 \end{vmatrix}.$$

Since $x_1^2 + x_2^2 = 1$ and $x_2 \neq 0$, this has the positive value n^2/x_2^2. This completes the proof.

The above proofs can be extended to the case where the coefficients of $f(x)$ belong to the Cayley algebra with 8 units, using mappings of an 8-dimensional sphere S_8 into itself. The following two facts should be kept in mind. (1) Due to the non-associativity of the Cayley numbers each monomial of $f(x)$ must be parenthesized in a definite manner, and a monomial of degree k may contain more than $k+1$ constants. (2) The monomial x^n need not be parenthesized.

The proof also applies equally well to the fundamental theorem of algebra in the usual case of complex variables. The resulting proof will use the degree of a mapping of the 2-sphere S_2 into itself. This proof differs slightly from the topological proof of Alexandroff and Hopf[5] which uses the degree of a mapping of S_1 into itself and the theorem of Rouché (in a topological formulation).

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[5] Loc. cit. p. 469. Cf. also Wei-Liang Chow, Math. Ann. vol. 116 (1939) p. 463.

ON ORTHOGONAL LATIN SQUARES

HENRY B. MANN[1]

An m-sided Latin square is an arrangement of the numbers $1, 2, \cdots, m$ into m rows and m columns in such a way that no row and no column contains any number twice. Two Latin squares are said to be orthogonal if when one is superimposed upon the other every ordered pair of numbers occurs once in the resulting square. Various methods have been devised for the construction of sets of orthogonal squares. However no method has as yet been given which would yield all possible sets of orthogonal Latin squares. In constructing orthogonal sets it is of value to have simple criteria which enable us to decide whether a given Latin square can be a member of an orthogonal pair.

A Latin square to which an orthogonal square exists will be called a basis square. In this note we shall derive two simple necessary conditions for a square to be a basis square.

THEOREM 1. *If in the Latin square L of side $4n+2$ the square formed by the first $2n+1$ rows and the first $2n+1$ columns contains fewer than $n+1$ numbers which are different from $1, 2, \cdots, 2n+1$ then L is not a basis square.*

PROOF. Denote by I the square formed by the first $2n+1$ rows and the first $2n+1$ columns, by II the square formed by the first $2n+1$ rows and the last $2n+1$ columns, by IV the square formed by the last $2n+1$ rows and the last $2n+1$ columns. Then if a number occurs a times in I it must occur $2n+1-a$ times in II and $2n+1-(2n+1-a)=a$ times in IV. Hence in I and IV together every number occurs $2a$ times. Assume now that I contains fewer than $n+1$ numbers different from $1, 2, \cdots, 2n+1$ and let L' be a square orthogonal to L. In the square resulting from superimposing L' on L every pair $1, i$ $2, i$ $\cdots$ $2n+1, i$ must occur. Hence every number i in L' occurs $2n+1$ times combined with a number of the set $1, 2, \cdots, 2n+1$ in L. But at most $2n$ numbers of the set $1, 2, \cdots, 2n+1$ occur in L outside of I and IV. Hence at least $2n+2$ numbers i of L' occur combined with the numbers $1, 2, \cdots, 2n+1$ in I and IV together an odd number of times. But at most $2n$ numbers of L' occur in I and IV combined with numbers of L which are different from $1, 2, \cdots, 2n+1$.

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Hence at least 2 numbers of L' would occur in I and IV an odd number of times; this is impossible.

COROLLARY 1. *If a $(4n+2)$-sided Latin square L contains in any subsquare formed by the rows $i_1, i_2, \cdots, i_{2n+1}$ and the columns $j_1, j_2, \cdots, j_{2n+1}$ fewer than $n+1$ numbers different from a given set of $2n+1$ numbers $k_1, k_2, \cdots, k_{2n+1}$, then L is not a basis square.*

Corollary 1 follows from Theorem 1 if we permute the rows, columns and numbers of L in a suitable manner. It may be remarked that using this corollary Tarry's [1][2] proof of the nonexistence of an orthogonal pair of 6-sided Latin squares can be simplified considerably.

We shall say that a Latin square L contains a Latin subsquare of side m if only m different numbers appear in a subsquare of L formed by m rows and m columns. From Corollary 1 we have another corollary:

COROLLARY 2. *If a $(4n+2)$-sided Latin square contains a subsquare of side $2n+1$ then it is not a basis square.*

Every multiplication table of a group forms a Latin square. A group of order $4n+2$ has a subgroup of order $2n+1$ which generates a Latin subsquare of side $2n+1$. We therefore have the following corollary:

COROLLARY 3. *A multiplication table of a group of order $4n+2$ is not a basis square.*

THEOREM 2. *If in the $(4n+1)$-sided Latin square L the square formed by the first $2n$ rows and the first $2n$ columns contains fewer than $n/2$ numbers different from $1, 2, \cdots, 2n$, then L is not a basis square.*

PROOF. Denote by I the square formed by the first $2n$ rows and the first $2n$ columns and by II the square formed by the last $2n+1$ rows and the last $2n+1$ columns. By an argument similar to that used in the proof of Theorem 1 it may be shown that if a number occurs a times in I it must occur $a+1$ times in II. Hence it occurs $2a+1$ times in I and II together. I and II together contain numbers different from $1, 2, \cdots, 2n$ at most $2n+1+n-1=3n$ times and outside of I and II numbers of the set $1, 2, \cdots, 2n$ occur at most $n-1$ times. If L' were orthogonal to L it would follow, using the methods of the proof of Theorem 1, that at least two numbers of L' would occur in I and II together an even number of times; this is impossible.

[2] Numbers in brackets refer to the References listed at the end of the paper.

COROLLARY 4. *If in the $(4n+1)$-sided Latin square L a subsquare formed by any $2n$ rows and any $2n$ columns contains fewer than $n/2$ numbers different from a given set $k_1, k_2, \cdots, k_{2n}$ of $2n$ numbers, then L is not a basis square.*

COROLLARY 5. *If a $(4n+1)$-sided Latin square L contains a $2n$-sided Latin subsquare, then L is not a basis square.*

In the following we shall use some notations and definitions introduced by the author in a previous paper [3].

Let P_i be the permutation which transforms $1, 2, \cdots, m$ into the ith row of the Latin square L. We identify L with its m row permutations and write $L=(P_1, P_2, \cdots, P_m)$. If $L=(P_1, \cdots, P_m)$, $L'=(P_1', P_2', \cdots, P_m')$ then we define $LL'=(P_1P_1', \cdots, P_mP_m')$. LL' is not necessarily a Latin square. From Theorem 1 of [3] it follows that two Latin squares A and B are orthogonal if and only if there exists a Latin square C such that $AC=B$. If P is any permutation and $A=(A_1, \cdots, A_m)$ a Latin square then we put $PA=(PA_1, \cdots, PA_m)$, $AP=(A_1P, \cdots, A_mP)$. PA and AP are also Latin squares. If $AC=B$ then $PAQQ^{-1}CR=PBR$. Hence we have the following lemma:

LEMMA 1. *If A is orthogonal to B then PAQ is orthogonal to PBR for any permutations P, Q, R.*

A is said to be equivalent to PAQ.

If A_1 is the identical permutation then A is said to be reduced. Clearly if A is reduced then also $P^{-1}AP$ is reduced.

We can now proceed to prove the following theorem.

THEOREM 3. *Every 5-sided basis square whose first row is 12345 is a multiplication table of the cyclic group of order 5.*

PROOF. It follows from Corollary 5 that a 5-sided basis square cannot contain a 2-sided Latin subsquare. Hence every row of a 5-sided basis square must be obtained from every other row by a cyclic permutation of order 5. Let $L=(1, P_2, P_3, P_4, P_5)$ be a basis square. P_2 is a cyclic permutation of order 5. Since in the symmetric group all cyclic permutations of the same order belong to the same class there exists a permutation Q such that $Q^{-1}P_2Q=(12345)$. $Q^{-1}LQ=L'$ is by Lemma 1 a basis square. Its first row is 12345 and its second row 23451. Considering that no 2-sided subsquare may be contained in a 5-sided basis square we obtain as possible third rows of L' the rows 31524, 35214, 34512. The first two of these third rows lead necessarily to Latin squares with 2-sided subsquares, the third leads

to a multiplication table of the group of order 5.

We shall say that a Latin square is based on a permutation group G if its rows can be obtained from the row $1, 2, \cdots, m$ by the permutations of G.

THEOREM 4. *Two Latin squares based on two different permutation groups of order 5 cannot be orthogonal.*

Let Z and Z' be two such groups. If $P \neq 1$ is in Z then $P^{-1}Z'P \neq Z'$, for otherwise one could obtain a group of degree 5 and order 25. By transforming Z' with the permutations of Z we can therefore obtain 5 different permutation groups all different from Z. But the symmetric group of degree 5 has only 6 subgroups of order 5. Hence every subgroup of order 5 which is different from Z can be obtained by transforming Z' with the permutations of Z. Such a transformation applied to Z itself leaves all elements of Z fixed. Hence if a Latin square based on Z were orthogonal to a Latin square based on Z' it would also be orthogonal to a Latin square based on any other group of order and degree 5.

It is therefore sufficient to show that the Latin square

$$Z = \begin{matrix} 12345 \\ 23451 \\ 34512 \\ 45123 \\ 51234 \end{matrix}$$

cannot be orthogonal to any Latin square L whose rows are 12345, 24531, 35214, 43152, 51423. By Lemma 1, Z is also orthogonal to LP_1^{-1} where P_1 is the first row permutation of L. Since the permutations of L form a group, LP_1^{-1} contains the same permutations as L. Thus it can always be achieved that the row 12345 stands over the first row of Z. Then since the pairs 11, 22, 33, 44, 55 all occur in the first row, the row 24531 must stand over the fourth row of Z and the row 35214 over the second row of Z. But then the pair 23 would occur twice. This proves Theorem 4.

In [3] the following definition was given. A biunique mapping S of a group G into itself is said to be a complete mapping if every element of G can be represented in the form XX^S where X is an element of G and X^S the image of X under the mapping S.

It was shown in [3] that two orthogonal Latin squares L_1, L_2 based on the same permutation group define a complete mapping S of the abstract group and that the squares can then be written as $L_1 = (P_1, \cdots, P_m), L_2 = (P_1P_1^S, \cdots, P_mP_m^S)$. If the mapping $T = 1 + S$,

that is to say the mapping defined by $X^T = XX^S$, is an automorphism then S is said to be derived from the automorphism T. In this case $L_1 = (P_1, \cdots, P_m)$, $L_2 = (P_1^T, \cdots, P_m^T)$.

We shall state this result as a lemma.

LEMMA 2. *If the orthogonal Latin squares L_1 and L_2 are based on the same permutation group G then $L_1 = (P_1, \cdots, P_m)$, $L_2 = (P_1P_1^S, \cdots, P_mP_m^S)$, where $P_1, \cdots, P_m$ are the permutations of G, and S is a complete mapping of G.*

In [3] it was shown: If S is a complete mapping of G and $P_1, \cdots, P_m$ is a regular representation of G then the Latin squares $(P_1, \cdots, P_m)$ and $(P_1P_1^S, \cdots, P_mP_m^S)$ are orthogonal.

We shall now define a slightly different procedure by which orthogonal squares may be obtained from complete mappings. It has been shown in [3] that every complete mapping can be transformed into a complete mapping S for which $1^S = 1$. In the following we shall consider only complete mappings which have this property.

Let $G_1 = 1, \cdots, G_m$ be the elements of a group G. Let S be a complete mapping of G. We form two Latin squares L_1 and L_2 in the following manner: In the ith row and kth column of L_1 write l if $G_iG_k = G_l$. In the ith row and kth column of L_2 write l if $G_iG_kG_k^S = G_lG_l^S$. L_1 and L_2 are Latin squares since they are multiplication tables of G. Moreover L_1 is orthogonal to L_2. Otherwise for some i, j, k, l with $(i, j) \neq (k, l)$ we should have

$$G_iG_j = G_kG_l = G_m \tag{1}$$

and

$$G_iG_jG_j^S = G_kG_lG_l^S = G_nG_n^S. \tag{2}$$

(Equations (1) and (2) must hold if the pair m, n occurs in the ith row and jth column and in the kth row and lth column.) But from (1) and (2) follows $G_j^S = G_l^S$.

Since S is a biunique mapping this is possible only if $j = l$. It follows then from (1) that $i = k$ contrary to our assumption.

Now let $L_1 = (P_1, \cdots, P_m)$, $L_2 = (P_1', \cdots, P_m')$. Let L_1 consist of the same permutations as L_2; we shall show that the mapping T defined by $G_i^T = G_iG_i^S$ is an automorphism.

To prove this we have to show that from

$$G_iG_i = G_k \tag{3}$$

follows

$$G_iG_i^SG_iG_i^S = G_kG_k^S. \tag{4}$$

By assumption $P_i=P_j'$ for some j. Hence if $G_iG_i=G_k$ we must have

$$G_jG_iG_i^S = G_kG_k^S. \tag{5}$$

P_i transforms 1 into l. Hence also P_j' transforms 1 into l and by our definition of L_2 we have

$$G_j11^S = G_jG_l^S. \tag{6}$$

But $11^S=1$. Hence $G_j=G_lG_l^S$ and (4) follows from (5). Hence we have the following theorem.

THEOREM 5. *If a group G admits a complete mapping which is not derived from automorphisms then there exist two orthogonal Latin squares based on two different permutation groups both isomorphic to G.*

COROLLARY 6. *The group of order 5 only admits complete mappings which are derived from automorphisms.*

The corollary is an immediate consequence of Theorems 4 and 5.

A set of $m-1$ orthogonal m-sided squares is said to be a complete set of Latin squares.

In the following we shall abbreviate finite two-dimensional projective geometry by PG_2.

In his beautiful paper [2] R. C. Bose established the connection between PG_2's and complete sets of Latin squares as follows.

Let G be a PG_2 with $m+1$ points on every line. We pick any line L of G and call it the line at infinity. Let $P_0, \cdots, P_m$ be the points of L. Through every one of these points pass m lines different from L. We number these lines and let L_{ij} $(i=0, \cdots, m; j=0, \cdots, m-1)$ denote the jth line passing through P_i. Every finite point P can then be identified with an $(m+1)$ad of numbers $(I_0, \cdots, I_m)$, where $I_j=k$ if L_{jk} passes through P. It is possible to identify every point with such an $(m+1)$ad of numbers since every pair of points determines exactly one line. A complete set of Latin squares can now be formed in the following way. Write the number k in the ith row and jth column of the square L_r $(r=2, \cdots, m)$ if k is the rth number of the $(m+1)$ad whose first number is i and whose second number is j. That is to say, we let the first two numbers be the row and column numbers. From the fact that two lines intersect in one and only one point it follows that $L_2, \cdots, L_m$ are orthogonal Latin squares.

Conversely if a complete set of Latin squares is given, R. C. Bose constructs a PG_2 in the following manner: Every "finite" point of the PG_2 is identified with an $(m+1)$ad of numbers $(I_0, \cdots, I_m)$

where, for $r \geqq 2$, the number I_r is the number in the I_0th row and the I_1st column of the rth Latin square. We form m^2+m lines L_{ik} ($i=0, \cdots, m$; $k=0, \cdots, m-1$), where L_{ik} is the set of all points whose ith number is k. Thus we obtain $m+1$ sets of parallel lines, each set containing m lines. From the orthogonality of the Latin squares it follows that two nonparallel lines intersect in only one point. We add now one point to each set of parallels and let this additional point lie on every line of the set. The set of added points forms the "line at infinity." Two lines then intersect in one and only one point. From this and the fact that $m+1$ lines pass through every point it follows easily that every pair of points is contained in one and only one line. Thus we obtain a PG_2 with m^2+m+1 points arranged in m^2+m+1 lines.

We shall exemplify R. C. Bose's construction by constructing a finite projective geometry consisting of 13 points from a set of 2 orthogonal 3-sided Latin squares. We start from a 3-sided Graeco-Latin square whose boxes, that is to say our finite points, we shall denote by letters. Thus we have

$11a$	$22b$	$33c$
$23d$	$31e$	$12f$
$32g$	$13h$	$21i$.

The sets of parallel lines are: Rows: *abc*, *def*, *ghi*. Columns: *adg*, *beh*, *cfi*. Numbers of the first Latin square: *afh*, *bdi*, *ceg*. Numbers of the second Latin square: *aei*, *bfg*, *cdh*.

Adding the "points at infinity" we obtain the finite geometry

abcj	*adgk*	*afhl*	*aeim*	*jklm*
defj	*behk*	*bdil*	*bfgm*	
ghij	*cfik*	*cegl*	*cdhm*	.

Let $0, 1, g_2, \cdots, g_{m-1}$ be the numbers of a Galois field. Form the Latin squares

$$L_j = \begin{matrix} 0 & 1 & \cdots & g_{m-1} \\ g_j & g_{j+1} & \cdots & g_j + g_{m-1} \\ g_j g_2 & g_j g_2 + 1 & \cdots & g_j g_2 + g_{m-1} \\ \cdot & \cdot & \cdots & \cdot \\ g_j g_{m-1} & g_j g_{m-1} + 1 & \cdots & g_j g_{m-1} + g_{m-1} \end{matrix} \qquad j = 1, 2, \cdots, m-1.$$

R. C. Bose [2] has shown that $L_1, \cdots, L_{m-1}$ form a complete set

of Latin squares. We shall say that this set is based on the Galois field $GF(m)$. We shall further say that a PG_2 is equivalent to a complete set S of Latin squares if one can be obtained from the other by Bose's method.

We shall prove the following theorem.

THEOREM 6. *If a PG_2 is equivalent to a complete set of Latin squares based on a Galois field then it is the analytic geometry of this Galois field.*

PROOF. The complete set of Latin squares can by hypothesis be represented by m^2 different $(m+1)$ads of numbers $(I_0, \cdots, I_m)$ of the Galois field, where $I_r=\alpha_r I_0+I_1$ $(r\geqq 1)$ with $\alpha_1=0$ and $\alpha_i\neq\alpha_j$ for $i\neq j$. The lines of the PG_2 consist of the points ($(m+1)$ads) with fixed I_r and of one point at infinity. Let $(1, I_0, I_1)$ be the coordinates (x, y, z) of every finite point and let the coordinates of the infinite point corresponding to the rth set of parallels be $(0, 1, -\alpha_r)$ for $r\geqq 2$, $(0, 1, 0)$ for $r=1$ and $(0, 0, 1)$ for $r=0$. Then all points on a line satisfy an equation $ax+by+cz=0$, namely $-I_r x+\alpha_r y+z=0$ for $r\geqq 1$ and $-I_0 x+y=0$ for the lines of the first set. Hence the PG_2 is the analytic geometry of $GF(m)$.

As an application of Theorems 3, 4, 5, and 6 we shall prove the following:

THEOREM 7. *Every PG_2 with 6 points on every line is the analytic geometry of $GF(5)$.*

PROOF. From Theorems 3, 4, and Corollary 6 it follows that every orthogonal pair L_1, L_2 of 5-sided reduced squares is based on the same permutation group. If $L_1=(1, P_2, \cdots, P_5)$ then, by Lemma 2, $L_2=(1, P_2P_2^S, \cdots, P_5P_5^S)$ where S is a complete mapping of the cyclic group of order 5. The mapping defined by $P_i^T=P_iP_i^S$ is by Corollary 6 an automorphism. Hence $L_2=(1, P_2^T, \cdots, P_5^T)$ where T is an automorphism. We write the group of order 5 as the additive group of remainders mod 5. Its only automorphisms are those induced by multiplication with the remainders 1, 2, 3, 4. Therefore if P_i is the permutation obtained by adding i to every remainder P_i^T must be the permutation obtained by adding αi to every remainder where α depends only on T. Hence every complete set of 5-sided Latin squares is based on the remainder system mod 5. Since the remainders mod 5 form a Galois field, Theorem 7 follows from Theorem 6.

It is also easy to show that every complete set of 2, 3, and 4-sided Latin squares is based on a Galois field. Hence also the uniqueness of PG_2's with 3, 4, and 5 points on every line can be proved in the same manner.

The uniqueness of finite geometries with less than 6 points on every line was first proved by J. H. M. Wedderburn and O. Veblen [4]. The uniqueness of finite geometries with 6 points on every line was first demonstrated by C. R. MacInnes [5] in a rather laborious tactical enumeration of cases.

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SOME THEOREMS ON CO-TERMINAL ARCS

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It is the purpose of this note to prove certain properties of sums of simple arcs which have one or both end points in common. The investigation was undertaken to answer a question, that of the validity of Theorem 3 below, raised by Miss Harlan C. Miller. An example is included to show that two of the results obtained are not valid for irreducible continua in general.

Theorem 1. *If H and K are two distinct arcs from A to B, then each point of $H+K-H\cdot K$ belongs to a simple closed curve lying in the closure of $H+K-H\cdot K$.*

Proof. Let P be any point of $H+K-H\cdot K=N$, and let S be the component of N which contains it. The set S is an arc segment; let its end points be X and Y. Suppose that no simple closed curve lying in $\overline{N}$ contains P. Then $\overline{N}-S$ contains no continuum containing both X and Y, for if it did it would contain an arc from X to Y, and this arc plus S would be a simple closed curve lying in $\overline{N}$ and contain-

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ing P. There exist, therefore, two disjoint closed sets U, containing X, and V, containing Y, such that $\overline{N}-S=U+V$. The components of $H+K-(U+V)$ are, except for S, subsets of $H\cdot K$ and hence are segments of arcs or segments plus one end point. Let U' be U plus all components of $H+K-(U+V)$ which have no end points in V, and let V' be V plus all components of $H+K-(U+V)$ which have no end points in U. Then U' and V' are closed and disjoint. Hence the components of $H+K-(U'+V')$ are finite in number since each of them has one end point in U' and the other in V'. Since all these segments but S are subsets of both H and K, while S is a subset of only one of these, it follows that the number of segments between U' and V' of one of the arcs H and K is even, while that of the other is odd. This is impossible since both H and K have A and B as end points.

THEOREM 2. *If H, K, and L are three distinct arcs from A to B, then there are two of them neither of which is a subset of the sum of the other two.*

PROOF. Suppose that H is a subset of $K+L$ and K is a subset of $H+L$; then $H+K-H\cdot K=N$ is a subset of L. This implies that $\overline{N}$ is a subset of L. By Theorem 1 there is a simple closed curve in $\overline{N}$. Hence the arc L contains a simple closed curve. It follows from this contradiction that one of the arcs H and K fails to be a subset of the sum of the other two. Suppose H does. Then it may be shown in the same way that one of the arcs K and L fails to be a subset of the sum of the other two, and the theorem is proven.

THEOREM 3. *If H, K, and L are three arcs emanating from a point A and no one of them is a subset of any other, then there is one of them which fails to be a subset of the sum of the other two.*

PROOF. Suppose that each of the arcs H, K, and L is a subset of the sum of the other two. Let the end points, other than A, of H, K, and L be B_H, B_K, and B_L, respectively. No two of the points B_H, B_K, and B_L coincide, for if they did a contradiction could be reached, as it was in the proof of Theorem 2, by the use of Theorem 1. If B_H did not belong to L, some neighborhood with respect to $H+K+L$ of B_H would be a subset of $H\cdot K$, which is impossible unless B_H and B_K coincide. Hence B_H and, similarly, B_K belong to L. Suppose that B_H precedes B_K in the order AB_L on L. Not every point of the interval AB_H of L belongs to H, or if it did H would be a subset of L. Hence there exists a segment S of the interval AB_L of L which contains no point of H. Let X and Y be the end points of S. Denote the intervals of AX and YB_L of L by U and V, respectively. A contradiction

may now be reached by an argument very similar to the last part of the proof of Theorem 1.

That Theorems 2 and 3 do not remain true if the word "arcs" in their hypotheses is replaced by the words "irreducible continua" is shown by the following example.

Example. There exist in the plane two points A and B and three continua, each irreducible from A to B, such that each of them is a subset of the sum of the other two.

Construction. Let C be a unit circle and S be C plus its interior. Denote by I_H, I_K, and I_L three mutually exclusive and non-abutting segments of C. For each positive integer n let D_{Hn}, D_{Kn}, and D_{Ln} be simple domains lying in $S-C$ such that if X is H, K, or L and $E_{Xn}=D_{X1}+D_{X2}+\cdots+D_{Xn}$, then (1) $\overline{D}_{X1}\cdot C=\overline{I}_X$, (2) $\overline{D}_{Xn}\cdot C=0$ if $n>1$, (3) $\overline{D}_{Xn}\cdot\overline{D}_{X(n+1)}$ is an arc of length less than $1/n$, (4) $\overline{D}_{Xn}\cdot\overline{D}_{X(n+i)}=0$ if $i>1$, and (5) each point of S is within $1/n$ units of E_{Xn}. Furthermore let D_{Xn} be such that $E_{Hn}\cdot E_{Kn}=E_{Kn}\cdot E_{Ln}=E_{Ln}\cdot E_{Hn}$. Then if $M_X=S-(I_X+D_{X1}+D_{X2}+\cdots)$, it follows that each of the sets M_H, M_K, and M_L is an indecomposable continuum,[1] that each of them is a subset of the sum of the other two, and that no one of them is a subset of any other. It is readily seen that there are two points between which each of these continua is irreducible.

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[1] It will be noted that the construction of the separate continua H, K, and L was made by the "canal" method. See K. Yoneyama, *Theory of continuous set of points*, Tôhoku Math. J. vol. 12 (1917) pp. 60–62.

COMPONENT ORBITS UNDER POINTWISE RECURRENT HOMEOMORPHISMS

MARIANO GARCÍA

1. Introduction. The notion of a component orbit is due to G. E. Schweigert,[1] and is closely associated with two theorems established by D. W. Hall and G. E. Schweigert,[2] but so far has been considered only in connection with pointwise periodic homeomorphisms. It is the purpose of this paper to consider an extended concept of this notion and, by applying methods analogous to those used by G. T. Whyburn[3] in obtaining the results of Hall and Schweigert, exhibit certain theorems that contain these results as special cases.

2. Preliminary definitions and remarks. Throughout this paper, X will denote a separable metric space. Let $f(X)=X$ be a homeomorphism defined on X and let $x\in X$.

The point x is *recurrent under* f provided that given any neighborhood U of x, there exists a positive integer n such that $f^n(x)\in U$.

The mapping f is *pointwise recurrent* on the set $A\subset X$ provided each point of A is recurrent under f. It should be noted that pointwise recurrence, as defined here, is equivalent to pointwise almost periodicity as used by Ayres, Whyburn, and others.

A non-empty invariant set L whose components can be ordered in a sequence

$$\cdots, A_{-2}, A_{-1}, A_0, A_1, A_2, \cdots$$

such that

$$f(A_i) = A_{i+1}, \qquad i = 0, \pm 1, \pm 2, \cdots,$$

will be termed a *component orbit*.

A component orbit one of whose components is invariant under a positive power of f will be termed a *periodic component orbit*.[4]

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[1] *A note on the limit of orbits*, Bull. Amer. Math. Soc. vol. 46 (1940) pp. 963–969.

[2] *Properties of invariant sets under pointwise periodic homeomorphisms*, Duke Math. J. vol. 4 (1938) pp. 719–724.

[3] *Analytic Topology*, Amer. Math. Soc. Colloquium Publications vol. 28 (1942) pp. 259–261. We shall refer to this book hereafter as ATW.

[4] Schweigert's component orbits become periodic component orbits under this definition.

By the *set-orbit* of a set $A \subset X$ we mean

$$\sum_{-\infty}^{+\infty} f^i(A),$$

and a set-orbit is *non-wandering*[5] if there exists a positive integer N such that $A \cdot f^N(A) \neq 0$.

It follows without difficulty that a component orbit is the set-orbit of a connected set, and conversely, the set-orbit of a connected set is a component orbit. Also, a periodic component orbit is the non-wandering set-orbit of a connected set, and conversely, the non-wandering set-orbit of a connected set is a periodic component orbit. Also, if a component orbit contains a periodic component orbit, it is a periodic component orbit, and in a compact space X, a component orbit is periodic if it is closed. Finally we remark that the orbit of a point x

$$\sum_{-\infty}^{+\infty} f^i(x)$$

is a component orbit, and if x is periodic, its orbit is a periodic component orbit.

3. **The main results.** We now prove the principal theorems.

THEOREM I. *Let $f(X) = X$ be a homeomorphism on a compact space X. If $G_1, G_2, \cdots$ is a sequence of component orbits whose limit inferior contains a periodic component orbit Q, and if both f and f^{-1} are pointwise recurrent on* $\limsup G_i - Q$, *then* $\limsup G_i$ *is a periodic component orbit.*

PROOF. Let Q_0 be a component of Q and let K_0 be the component of $L = \limsup G_i$ which contains Q_0. Since L is invariant, the image of K_0 under any power of f is a component of L. Let $f(K_0) = K_1, f(K_1) = K_2, \cdots, f(K_{r-1}) = K_r, f(K_r) = K_0$, so that

$$K = \sum_{0}^{r} K_i$$

is a periodic component orbit which contains Q. We shall show $K = L$.

Suppose this is not true. Then we can find[6] an open set V containing K, whose boundary $F(V)$ does not intersect L and such that L is not contained wholly in V.

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[5] In this connection see G. D. Birkhoff and P. A. Smith, *Structure analysis of surface transformations*, J. Math. Pures Appl. vol. 7 (1928) p. 350.

[6] ATW, Theorem (9.3), p. 15. The proof of Theorem I follows closely the proof of Theorem (6.5), p. 259.

Select a point x_i from each G_i such that $x_i \to x \in Q_0$. Infinitely many G_i intersect $X - \overline{V}$ and thus we may assume that for each i there exists a point $y_i \in G_i$ such that $y_i \in X - \overline{V}$. Let C_{z_i} be the component of G_i containing x_i. Since $x \in \liminf C_{z_i}$, $\limsup C_{z_i}$ is connected[7] and hence is contained in K_0. Also,[8] for almost all i, and therefore we may assume for all i, $C_{z_i} \subset V$. Each C_{z_i} contains a point z_i of the orbit of y_i. A subsequence of $\{z_i\}$, which we may suppose is the whole sequence, converges to a point $z \in K_0$.

There exists a subsequence $\{z_{i_j}\}$ of $\{z_i\}$ such that for each i_j, y_{i_j} is a positive power of z_{i_j} or for each i_j, y_{i_j} is a negative power of z_{i_j}. We shall assume that this sequence is $\{z_i\}$ and that y_i is a negative power of z_i for each i, since the proof for the case where y_i is a positive power of z_i for each i is perfectly analogous.

Now, let b_i be the first point in the sequence

$$z_i, f^{-1}(z_i), f^{-2}(z_i), \cdots$$

which is in $X - V$. A subsequence of $\{b_i\}$, which again we may suppose is the whole sequence, converges to a point $b \in L$. Furthermore, $b \notin \overline{V}$, since $L \cdot F(V) = 0$.

For each i, $f(b_i)$ is in V and consequently

$$f(b_i) \to f(b) \in V \cdot (L - K),$$

since $L - K$ is invariant.

Now, there exists an integer N_1 such that

$$f(b_i) \neq z_i \qquad \text{for} \quad i > N_1.$$

This is true because otherwise we should have

$$f(b_i) \to z$$

and therefore

$$b_i \to f^{-1}(z) \neq b.$$

Thus the points $f(b_i)$, $i > N_1$, are negative powers of z_i which are in V, and therefore the points $f^2(b_i)$, $i > N_1$, are in V. Also,

$$f^2(b_i) \to f^2(b) \in V \cdot (L - K).$$

But again, there exists an integer $N_2 > N_1$ such that

$$f^2(b_i) \neq z_i \qquad \text{for} \quad i > N_2.$$

The points $f^2(b_i)$, $i > N_2$, are then in V and

[7] ATW, Theorem (9.1), pp. 14–15.
[8] ATW, Theorem (7.2), p. 12.

$$f^{i}(b_i) \to f^{i}(b) \in V \cdot (L - K).$$

But we can find an integer $N_3 > N_2$ such that

$$f^{i}(b_i) \neq z_i \qquad \text{for} \quad i > N_3,$$

and continuing in this manner we obtain

$$f^{n}(b) \in V, \qquad n = 1, 2, 3, \cdots.$$

This contradicts the recurrence of the point b under f, since $b \notin \overline{V}$ and therefore we can find a neighborhood R of b such that

$$f^{n}(b) \cdot R = 0, \qquad n = 1, 2, 3, \cdots.$$

Had we considered a sequence $\{z_i\}$ for which y_i is a positive power of z_i for each i, we would have obtained a contradiction to the recurrence under f^{-1} of some point of lim sup $G_i - Q$.

THEOREM II. *Let $f(X) = X$ be a homeomorphism on a compact space X. If $G_1, G_2, \cdots$ is a sequence of component orbits whose limit inferior contains a periodic component orbit Q and if f is pointwise recurrent on* lim sup $G_i + \sum_{i=1}^{+\infty} G_i$, *then* lim sup G_i *is a periodic component orbit.*

PROOF. This follows from the proof of the preceding theorem. Let O_i be the orbit determined by y_i and z_i in the proof of Theorem I. There exists a neighborhood U_i of z_i, of diameter less than $1/i$, such that $U_i \subset V$ and, due to the recurrence of z_i under f, corresponding to U_i we can find a point $r_i \in U_i$ such that r_i is a positive power of y_i. Let b_i be the first point in the sequence

$$r_i, f^{-1}(r_i), f^{-2}(r_i), \cdots$$

which is in $X - V$. Since $r_i \to z$, we can proceed as in the proof of Theorem I to show that the point b, to which we may assume $\{b_i\}$ converges, is not recurrent under f.

From the previous proofs we immediately have the following corollary.

COROLLARY 1. *Under the hypotheses of Theorem* I *or of Theorem* II, *the number of components of* lim sup G_i *can not exceed the number of components of Q. Thus, if* lim inf G_i *contains an invariant connected set (in particular, if it contains a fixed point), then* lim sup G_i *is connected.*

COROLLARY 2. *Let $f(X) = X$ be a homeomorphism on a compact space X. Let G be a component orbit in X. If f is pointwise recurrent on $\overline{G}$ and $\overline{G}$ contains a periodic component orbit Q, then $\overline{G}$ is a periodic component orbit and the number of its components does not exceed the number of components of Q.*

PROOF. This is a direct consequence of Theorem II and Corollary 1, for we need only take $G_i = G$ for all i. Then $\lim G_i = \overline{G}$ and the conclusion follows.

COROLLARY 3. *Any pointwise periodic continuous mapping g of a compact totally disconnected space X into itself has equicontinuous powers.*[9]

PROOF. It is a well known fact[10] that under the above hypotheses g is a homeomorphism of X onto itself. Let us show that the orbit decomposition given by g is continuous.[11]

Let (O_n) be a sequence of orbits of X such that $x \in \lim\inf (O_n)$. Then, letting O_x be the orbit of x,

$$O_x \subset \lim\inf (O_n) \subset \lim\sup (O_n).$$

Now, by Theorem I, $\lim\sup (O_n)$ is a periodic component orbit and, since X is totally disconnected, $\lim\sup (O_n)$ is an orbit.

Thus, $\lim (O_n) = O_x$ and therefore the orbit decomposition is continuous, from which it follows that f has equicontinuous powers.[12]

THEOREM III. *Let $f(X) = X$ be a homeomorphism on a compact space X. If $L = L_1 + L_2$ is a separation of a non-empty, closed, invariant set L in X such that f is pointwise recurrent on L and, for $i = 1$ or for $i = 2$, L_i has the property that the closure of the orbit of each of its points is a periodic component orbit,*[13] *then there exists a positive integer N such that*

$$f^N(L_i) = L_i, \qquad i = 1, 2.$$

PROOF. It is no restriction to assume that L_1 has the indicated property: $x \in L_1$ implies that the closure of the orbit of x is a periodic component orbit.

Since f is pointwise recurrent[14] on L, if the theorem is false, for each n there exists an $x_n \in L_1$ such that $f^{n!}(x_n) \in L_2$.

Choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converging to a point $x \in L_1$. Let O be the orbit of x. By hypothesis,

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[9] For the meaning of the terms involved in this corollary, see ATW, pp. 34, 239, 252.

[10] ATW, Theorem (1.1) and Corollary (1.11), p. 240.

[11] ATW, pp. 129–130.

[12] ATW, Theorem (6.4), p. 257. See also Corollary (6.42), p. 258.

[13] We could substitute "contains a periodic component orbit" for the phrase "is a component orbit," but in virtue of Corollary 2 this does not matter.

[14] In connection with the phrase that follows, see ATW, Theorem (4.12), p. 247, keeping in mind that "pointwise almost periodic," as used there, is "pointwise recurrent" in our notation.

$$\overline{O} = A_0 + A_1 + \cdots + A_k,$$

where A_0 is a component of $\overline{O}$ and

$$f(A_i) = A_{i+1}, \qquad i = 0, 1, \cdots, k-1,$$

and $f(A_k) = A_0$. We may also assume that $A_r \neq A_s$ for $r \neq s$, r and s being any integers, $0 \leqq r, s \leqq k$.

Then $x \in A_j$, where j is an integer such that $0 \leqq j \leqq k$.

Define $g(y) = f^k(y)$, $y \in L$. By a result of W. H. Gottschalk,[15] g is pointwise recurrent on L.

For each n_i, let G_{n_i} be the orbit under g of x_{n_i}. Now,

$$x \in \liminf G_{n_i}.$$

Furthermore, since

$$\sum_{i=-\infty}^{+\infty} f^{ki}(x) \subset \liminf G_{n_i},$$

$$A_j = \overline{\sum_{i=-\infty}^{+\infty} f^{ki}(x)} \subset \liminf G_{n_i}.$$

But now $g(L) = L$ is a homeomorphism on the compact set L and $\{G_{n_i}\}$ is a sequence of orbits whose limit inferior contains the connected set A_j. Also, A_j is invariant under g and g is pointwise recurrent on L. Thus, by Theorem II, lim sup G_{n_i} is connected and consequently

$$\limsup G_{n_i} \subset L_1,$$

since $x \in L_1$.

But for each i,

$$f^{n_i+1}(x_{n_i}) \in L_2$$

and for $n_i > k$,

$$f^{n_i+1}(x_{n_i}) = g^{n_i+1/k}(x_{n_i}).$$

Thus it follows that $G_{n_i} \cdot L_2 \neq 0$ for $n_i > k$. This gives

$$L_2 \cdot \limsup G_{n_i} \neq 0,$$

which contradicts lim sup $G_{n_i} \subset L_1$.

4. **Conclusion.** The main results that Hall and Schweigert obtained relative to component orbits can be stated in the following way:[16]

[15] *Powers of homeomorphisms with almost periodic properties*, Bull. Amer. Math. Soc. vol. 50 (1944) Corollary 1, p. 223. This result is very useful here.

[16] See footnote 3.

THEOREM 1. *Let $f(X)=X$ be a pointwise periodic homeomorphism on a compact space X. Then if $G_1, G_2, \cdots$ is a convergent sequence of periodic component orbits in X, the limit set L is a periodic component orbit.*

THEOREM 2. *Let X be compact and suppose $f(X)=X$ is pointwise periodic. Any non-empty disconnected invariant closed set L in X has the property that for every separation*

$$L = L_1 + L_2$$

there exists an integer N such that

$$f^N(L_i) = L_i, \qquad i = 1, 2.$$

Theorem 1 follows immediately from either of Theorems I and II of this paper, for since f is pointwise periodic on X, f and f^{-1} are pointwise recurrent on X. Also, since X is compact, any set which is the limit of a convergent sequence of component orbits is non-empty, and therefore contains a periodic orbit, which clearly is a periodic component orbit. Thus, the hypotheses of Theorems I and II are satisfied and hence $\lim G_i$ is a periodic component orbit. In this connection it may be remarked that a homeomorphism $f(X)=X$ is pointwise periodic if and only if every component orbit in X is a periodic component orbit, where X is not necessarily compact.

That Theorem I is considerably more general than Theorem 1 is evident in more than one sense. Referring to the notation employed in Theorem I, examples satisfying the hypotheses of the theorem can be constructed for which f is not pointwise recurrent on Q and such that f is not pointwise periodic on $\limsup G_i - Q$.

Theorem 2, which incidentally may be used to prove Corollary 3 of this paper, is a direct consequence of Theorem III. For if f is pointwise periodic on X, the closure of any orbit in X is a periodic component orbit; namely, the orbit itself.

Here again Theorem III is of more generality than Theorem 2. Using the notation of Theorem III, examples fulfilling the conditions of the theorem can be given for which there exist points in L_2, say, the closures of whose orbits are not even component orbits, and such that f is not pointwise periodic on L_1 nor on L_2.

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GROUPS WITHOUT PROPER ISOMORPHIC QUOTIENT GROUPS

REINHOLD BAER

If f is a homomorphism of the group G, and if g is an isomorphism of the image group G', then fg is a homomorphism of G too; and this homomorphism is an isomorphism if, and only if, f is an isomorphism. Consequently the following three properties of the group G imply each other.

(1) *Homomorphisms of G upon isomorphic groups are isomorphisms.*

(2) *Homomorphisms of G upon itself are isomorphisms.*

(3) *If N is a normal subgroup of G such that G and G/N are isomorphic, then $N=1$.*

Groups meeting these requirements (1) to (3) shall be termed Q-groups. A direct product of an infinity of isomorphic groups different from 1 is certainly not a Q-group. On the other hand it is readily seen that groups satisfying the ascending chain condition for normal subgroups are Q-groups. Deeper is the fact that the group G is a Q-group if it belongs to one of the following three classes of groups.

(a) Free groups on a finite number of generators.[1]

(b) Groups of finite dimensional matrices with coefficients from a field, which are generated by a finite number of elements.[2]

(c) Free products of a finite number of abelian groups each of which is generated by a finite number of elements.[3]

All these groups are generated by a finite number of elements. But we are able to show that this latter condition is neither necessary nor sufficient for a group to be a Q-group. The complexity of the situation is increased by the fact that neither subgroups nor quotient groups of Q-groups need be Q-groups. Thus it becomes desirable to obtain general criteria for a group to be a Q-group, and this is the main object of the present note.

The subgroup S of the group G shall be termed *completely characteristic, if $S=S'$ for every homomorphism f of G which satisfies $G=G'$.* Examples of completely characteristic subgroups are the commutator subgroup of G and its generalizations. If G happens to be a Q-group, then every characteristic subgroup of G is completely characteristic.

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[1] Levi [1]; Magnus [2]. The numbers in brackets refer to the bibliography at the end of the paper.

[2] Malcew [1].

[3] Fouxe-Rabinowitch [1].

THEOREM 1. *The group G is a Q-group if there exists a well ordered ascending chain of completely characteristic subgroups S_v with the following properties.*

(i) $S_0=1$; $S_k=G$ *for some* k.

(ii) S_{v+1}/S_v *is a Q-group for every* $v<k$.

(iii) *If v is a limit ordinal, then every element in S_v is contained in some S_u with $u<v$.*

PROOF. Suppose that f is a homomorphism of G satisfying $G=G^f$, and that N is the kernel[4] of this homomorphism f. We are going to prove by complete (transfinite) induction that the crosscut[5] of N and S_v is 1. Since this assertion is certainly true for $v=0$, we may assume its validity for every $u<v$.

Case 1. $v=u+1$ is not a limit ordinal.

Then we infer from our induction hypothesis that $N\cap S_u=1$. We deduce furthermore $S_v=S_v^f$ and $S_u=S_u^f$ from the fact that the subgroups S are completely characteristic subgroups of G and that $G=G^f$. Thus f induces a homomorphism of S_v/S_u such that $S_v/S_u=(S_v/S_u)^f$. But S_v/S_u is by (ii) a Q-group and f consequently induces an isomorphism in S_v/S_u. If x is an element in the crosscut of N and S_v, then $(S_ux)^f=S_u$. Hence $S_ux=S_u$ and x is in S_u. But the crosscut of S_u and N is 1, proving $x=1$. Thus $N\cap S_v=1$.

Case 2. v is a limit ordinal.

If x belongs to $S_v\cap N$, then x belongs by (iii) to $S_u\cap N$ for $u<v$. Hence $x=1$ may be inferred from the induction hypothesis, showing again that $N\cap S_v=1$.

Thus it follows in particular that $N=N\cap G=N\cap S_k=1$, proving that f is an isomorphism, as we desired to show.

The subgroup S of the group G has been termed *strictly characteristic*,[6] *if $S^f\leqq S$ for every homomorphism f of G satisfying $G^f=G$.* The members $Z_v(G)$ of the *ascending central chain* of the group G are strictly characteristic subgroups of G. To prove this it suffices to recall their inductive definition.

(i) $Z_0(G)=1$.

(ii) $Z_{v+1}(G)/Z_v(G)$ is the center[7] of $G/Z_v(G)$.

(iii) If v is a limit ordinal, then $Z_v(G)$ is the (set theoretical) join of the subgroups $Z_u(G)$ for $u<v$.

[4] The kernel of the homomorphism f consists of the elements in G which are mapped upon 1 by f.

[5] The crosscut of the sets A and B will always be denoted by $A\cap B$.

[6] For a discussion of this and related concepts see Baer [2].

[7] The center of the group G consists of all the elements z in G which satisfy $zx=xz$ for every x in G.

Let us note that the group G is termed *nilpotent*[8] if there exists an ordinal z such that $G=Z_z(G)$.

The members of the ascending central series, though strictly characteristic, need not be completely characteristic. This will be apparent from the following example,[9] which shows at the same time the impossibility of substituting "strictly characteristic" for "completely characteristic" in Theorem 1.

Let B be an abelian group of type p^∞. Then B contains one and only one subgroup B_i which is a cyclic group of order p^i, and every element in B is in at least one B_i. Denote by G the group obtained by adjoining to B an element g subject to the relation

$$g^{-1}xg = x^{1+p}$$

for x in B. It is readily seen that $B_i=Z_i(G)$ for finite i, that $B=Z_\omega(G)$ and that $G=Z_{\omega+1}(G)$. Thus G is nilpotent and the quotient groups of successive terms in the ascending central series are cyclic groups (and therefore Q-groups). But G is not a Q-group, since G and G/B_1 are clearly isomorphic groups.

COROLLARY. *If G is a nilpotent group and if every $Z_{\nu+1}(G)/Z_\nu(G)$ is a Q-group, then the following condition is necessary and sufficient for G to be a Q-group.*

Every $Z_\nu(G)$ is a completely characteristic subgroup of G.

The sufficiency of this condition is an immediate consequence of Theorem 1; its necessity may be derived from the fact that every characteristic subgroup of a Q-group is completely characteristic. The impossibility of omitting this condition from the statement of this corollary is a consequence of the considerations immediately preceding this corollary.

LEMMA. *If S is a strictly characteristic subgroup of the group G, and if G/S is a Q-group, then S is completely characteristic.*

PROOF. If f is an endomorphism of G such that $G=G'$, then $S'\leqq S$ and an endomorphism of G/S upon itself is defined by mapping the coset X in G/S upon the coset SX' in G/S. Since G/S is a Q-group, it follows that this endomorphism of G/S is an automorphism. From this fact we conclude that the element x in G belongs to S if, and only if, x' belongs to S. But to every element y in S there exists an element z in G such that $z'=y$; and from the preceding remark we infer now that z belongs to S too, proving $S=S'$, as we desired to show.

[8] See Baer [1] for a discussion of this concept of nilpotency.

[9] This is essentially a restatement of Baer [1, Example 2.5, p. 406].

The next criterion is an almost exact dual to Theorem 1 with the important difference that we need only consider strictly characteristic subgroups.

THEOREM 2. *The group G is a Q-group if there exists a well ordered descending chain of strictly characteristic subgroups S_v with the following properties.*

(i) $S_0=G$; $S_k=1$ *for some ordinal* k.

(ii) S_v/S_{v+1} *is a Q-group for every* $v<k$.

(iii) *If v is a limit ordinal, then S_v is the crosscut of the subgroups S_u for* $u<v$.

PROOF. If the homomorphism f of the group G satisfies $G=G^f$, then we are going to prove the following assertion by complete (transfinite) induction with respect to v.

(v) $S_v=S_v^f$ *and f induces an isomorphism of* G/S_v.

Since this assertion (u) certainly holds true for $u=0$, we may assume its validity for every $u<v$, in order to prove (v). We distinguish two cases.

Case 1. $v=u+1$ is not a limit ordinal.

Then we infer from the induction hypothesis that $S_u=S_u^f$ and that f induces an isomorphism in G/S_u. Since S_v is strictly characteristic, we have $S_v^f\leqq S_v$. If X is a coset in S_u/S_v, then mapping X upon S_vX^f effects a homomorphism of S_u/S_v upon itself, that is, S_u/S_v is the image of S_u/S_v under this homomorphism. But S_u/S_v is by (ii) a Q-group, and thus f induces a proper automorphism of S_u/S_v. Suppose now that X is a coset of G/S_v which is mapped upon 1 by the homomorphism f of G/S_v. Then $X^f\leqq S_v$ and therefore $S_uX^f=S_u$. But f induces an isomorphism in G/S_u so that $X\leqq S_u$ and X is a coset of S_u/S_v. But then $X=1$, since f induces an isomorphism in S_u/S_v, and we have shown that f induces a proper automorphism of G/S_v. If x is an element in S_v, then there exists an element y in G such that $x=y^f$. The coset S_vy is mapped by f into part of S_v. Since f induces an isomorphism in G/S_v, as we have shown just now, it follows that $S_vy=S_v$, and this proves $S_v=S_v^f$, completing the proof of the case $(v)=(u+1)$.

Case 2. v is a limit ordinal.

If X is a coset of G/S_v such that $X^f\leqq S_v$, then $X^f\leqq S_u$ for $u<v$. From the induction hypothesis we infer that f induces an isomorphism in G/S_u for $u<v$. Since f maps the coset S_uX of G/S_u upon the identity of G/S_u, it follows that $X\leqq S_u$ for $u<v$, and we infer from (iii) that $X\leqq S_v$. Thus f induces a proper automorphism of G/S_v; the fact that $S_v=S_v^f$ is verified as in Case 1.

Thus we have verified (v) for every $v \leqq k$. From condition (i) and proposition (k) we infer finally that f is an automorphism of G, as we desired to show.

To enunciate two important consequences of this theorem we need the *descending central series* of G, whose members ${}^{\nu}G$ are defined inductively as follows:

(i) ${}^{0}G = G$.

(ii) ${}^{\nu+1}G$ is generated by all the commutators $x^{-1}y^{-1}xy$ for x in G and y in ${}^{\nu}G$.

(iii) If v is a limit ordinal, then ${}^{v}G$ is the crosscut of all the ${}^{u}G$ for $u < v$.

It is readily verified that ${}^{\nu}G' \leqq {}^{\nu}G$ for every endomorphism[10] f of G; that is, the subgroups ${}^{\nu}G$ are fully invariant,[11] and they are both completely and strictly characteristic subgroups of G. This descending central chain certainly meets the requirement (iii) of Theorem 2, and thus we obtain the following criterion.

COROLLARY 1. *The group G is a Q-group, if there exists an ordinal k such that ${}^{k}G = 1$, and if the (abelian) groups ${}^{\nu}G/{}^{\nu+1}G$ are all Q-groups.*

An abelian group is certainly a *Q-group* if it is generated by a finite number of elements. It is well known furthermore that ${}^{i}G/{}^{i+1}G$, for finite i, is generated by a finite number of elements if G is generated by a finite number of elements. Thus the following fact is a special case of Corollary 1.

COROLLARY 2.[12] *The group G is a Q-group if it is generated by a finite number of elements, and if ${}^{\omega}G = 1$.*

If F is a free group, then Magnus[13] has shown that ${}^{\omega}F = 1$, and thus the last criteria are generalizations of the theorem[14] that free groups on a finite number of generators are Q-groups.

In order to apply Theorem 2 to abelian groups we introduce some notations. If A is an abelian group, then denote by $F_p(A)$ the set of all the elements in A whose order is a power of the prime number p. Clearly $F_p(A)$ is a fully invariant subgroup of A. The product $F(A)$ of all the $F_p(A)$ is their direct product and consists of all the elements of finite order whereas $A/F(A)$ contains no elements of finite order except 1.

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[10] Endomorphisms of the group G are homomorphisms of G into G.

[11] This concept is due to Levi [1]; for further discussion see Baer [2].

[12] Fouxe-Rabinowitch [1].

[13] Magnus [1].

[14] See footnote 1.

COROLLARY 3. *If $A/F(A)$ as well as every $F_p(A)$ is a Q-group, then the abelian group A is a Q-group.*

PROOF. Let $S_0=A$, $S_1=F(A)$, $S_i=\prod_{i\leq j}F_{p_j}(A)$ where $p_1, p_2, \cdots$ is some enumeration of the primes. The subgroups S_i are clearly strictly characteristic and their crosscut $S_\omega=1$. Since $S_0/S_1=A/F(A)$ and since S_i/S_{i+1} is, for positive i, isomorphic to $F_{p_i}(A)$, the hypotheses of Theorem 2 are satisfied by this descending chain of subgroups, proving the validity of our present contention.

Let us note that $F_p(A)$ is a Q-group if it is finite, and that $F/F(A)$ is a Q-group if it is of finite rank, that is, if there exists an integer k such that every subgroup of $F/F(A)$ which may be generated by a finite number of elements may also be generated by k elements. Combining Corollaries 1 and 3 and the preceding remarks it is easy to construct any number of examples of Q-groups that cannot be generated by a finite number of elements.

Examples. 1. It is known that the free group on two generators contains a free group on an infinity of generators as a subgroup.[15] The former group is a Q-group, the latter is not. This shows that *not every subgroup of a Q-group need be a Q-group.*

2. The additive group R of the rational numbers cannot be generated by a finite number of elements. But R is a Q-group. For if R is considered modulo a subgroup different from 0, then all the elements in the quotient group are of finite order. If J is the subgroup of the integers in R, and if E is the subgroup of the even integers in R, then $E\leqq J$ and R/J and R/E are isomorphic groups so that R/E is not a Q-group, showing that *quotient groups of Q-groups need not be Q-groups.*

3. We are going to construct an example of a group G with the following properties.

(a) *G is generated by two elements.*

(b) $(G, G'')=1$.

(c) *G is not a Q-group.*

Here we denote by G' the commutator subgroup of G, by G'' the commutator subgroup of G', and by (G, G'') the subgroup generated by all the commutators $x^{-1}y^{-1}xy$ for x in G and y in G''.

This example shows that Corollary 2 of Theorem 2 is in certain respects a "best" result.

To construct the group G we proceed as follows. Let F be a free group on two generators and denote by u, v a free pair of generators of F. Then F/F' is the free abelian group of rank 2. The group F'/F''

[15] For example, the commutator subgroup.

may be shown to be a free abelian group, a basis of which is given by the elements

$$(i, j) = u^{-i}v^{-j}u^{-1}v^{-1}uv\,v^{j}u^{i}$$

for integral i, j. We put $N=(F, F'')$. Then F''/N is part of the center of F/N; it is a free abelian group, generated by the elements (commutators)

$$(i, j)(i', j')(i, j)^{-1}(i', j')^{-1}.$$

Every subgroup of F''/N is a normal subgroup of F/N, since F''/N is part of the center of F/N. We denote by U the subgroup of F generated by N and the elements $(0, j)(0, j')(0, j)^{-1}(0, j')^{-1}$ for all positive j, j', and we denote by V the subgroup of F generated by N and all the elements $(0, j)(0, j')(0, j)^{-1}(0, j')^{-1}$ for j, j' not negative. It is clear that U and V are normal subgroups of F and that

$$N < V < U < F''.$$

We are going to prove now that $G=F/V$ is the desired group. It is immediately clear that conditions (a) and (b) are satisfied by G. To prove (c) we construct an automorphism of the group F which maps V upon U. There exists one and only one automorphism f of F which maps u upon uv and v upon v. We have

$$F'f = F', \qquad F''f = F'', \qquad Nf = N,$$

since these subgroups are characteristic subgroups of F. Furthermore we have

$$(0, 0)f = (u^{-1}v^{-1}uv)f = (uv)^{-1}v^{-1}(uv)v = v^{-1}(0, 0)v = (0, 1),$$

and consequently

$$(0, j)f = (0, j + 1)$$

for integral j. From this last equation one derives the equality $Vf = U$, completing the proof.

From the preceding arguments one deduces readily the noteworthy fact that the ascending chain condition is not satisfied by the normal subgroups of a nonabelian free group.

Dualizing the concept of a Q-group we define as an *S-group* a group G meeting the following requirement.

(*S*) *If the subgroup S of G is isomorphic to G, then $S=G$.*

It is readily seen that every group satisfying the descending chain condition for subgroups is an S-group. An example of an S-group not satisfying this chain condition is the additive group of rational numbers.

To enunciate conveniently a criterion for S-groups we need the following concept. The subgroup C of the group G is said to be an *S-characteristic subgroup of G* if $C^f \leqq C$ for every isomorphism f of G into itself. Clearly every fully invariant subgroup is S-characteristic and every S-characteristic subgroup is characteristic. If G happens to be an S-group, then conversely every characteristic subgroup of G is S-characteristic.

THEOREM 3. *The group G is an S-group if there exists a well ordered ascending chain of S-characteristic subgroups $N(v)$ of G with the following properties.*

(i) $N(0)=1$ *and* $N(t)=G$ *for some ordinal* t.

(ii) $N(v+1)/N(v)$ *is an S-group.*

(iii) *If v is a limit ordinal, then every element in $N(v)$ is contained in some $N(u)$ for $u<v$.*

PROOF. Suppose that f is an isomorphism of G upon some subgroup S of G. We are going to prove by complete (transfinite) induction that $N(v)^f=N(v)$ for every v. This assertion is certainly true for $v=0$; and thus we may assume its validity for every $u<v$. If v happens to be a limit ordinal, then our contention is directly derived from (iii) and the induction hypothesis. If, however, $v=u+1$ is not a limit ordinal, then we deduce from $N(u)^f=N(u)$ and $N(v)^f \leqq N(v)$ that f induces an isomorphism of the group $N(v)/N(u)$ upon a subgroup of $N(v)/N(u)$. But it follows from (ii) that $N(v)/N(u)$ is an S-group. Hence $(N(v)/N(u))^f=N(v)/N(u)$ or $N(v)=N(v)^fN(u)=N(v)^fN(u)^f=N(v)^f$, as we desired to show. Now $G=N(t)=N(t)^f=G^f$ is a consequence of (i).

We are interested in the relations between Q-groups and S-groups. There are groups that are both, like the finite groups or the additive group of rational numbers. Any direct (or free) product of an infinity of isomorphic groups ($\neq 1$) is neither. The infinite cyclic groups are Q-groups, but not S-groups, and the abelian groups of type p^∞ are S-groups, but not Q-groups. Thus the following theorem seems to be of some interest.

THEOREM 4. *If N is a fully invariant subgroup of the free group F, and if $G=F/N$ is an S-group, then G is a Q-group.*

PROOF. If L is a normal subgroup of G such that G and G/L are isomorphic groups, then denote by M the uniquely determined normal subgroup of F such that $N \leqq M$ and $M/N=L$. There exists an isomorphism g of G upon G/L, and hence there exists an endomorphism f of F satisfying $X^g=MX^f$ for every coset X of $F/N=G$. (To

prove the existence of f, consider a free set of generators of F and define f on this free set of generators in accordance with our requirement. Then f can easily be seen to be a desired endomorphism.) Then $F = F^fM$ and $F^f \cap M = N^f \leqq N$, since N is a fully invariant subgroup of F. Since M and N are normal subgroups of F, we deduce from Dedekind's law that

$$F^fN \cap M = (F^f \cap M)N = N^fN = N,$$

and from $N \leqq M$ we infer $(F^fN)M = F^fM = F$. The subgroup $(F^fN)/N = V$ of $G = F/N$ has therefore the following properties:

$$V \cap L = 1 \quad \text{and} \quad VL = G.$$

Thus every coset of G/L contains one and only one element of the subgroup V of G, proving that V and G/L are isomorphic. But G/L and G are isomorphic and G is an S-group, proving that $V = G$. This implies finally $L = 1$, showing that G is a Q-group.

It is apparent from the proof of this theorem that it remains valid if we substitute for the hypothesis that F be a free group the following assumption:

If M and N are normal subgroups of F, if $N \leqq M$, and if g is an isomorphism of F/N upon F/M, then there exists an endomorphism f of F, satisfying

$$X^g = MX^f$$

for every coset X of F/N.

It has been shown that the properties (1) to (3) may be deduced (a) from the existence of an ascending chain of completely characteristic subgroups, meeting certain requirements (Theorem 1), and (b) from the existence of a descending chain of strictly characteristic subgroups, satisfying certain conditions (Theorem 2). But these properties (a) and (b) are not independent. For the following correspondence between subgroups maps every ascending chain of completely characteristic subgroups upon a descending chain of strictly characteristic subgroups. If S is a subgroup of the group G, then let S^* be *the centralizer of S in G*, which consists exactly of those elements in G commuting with all the elements in S. To substantiate our contention we enumerate a number of properties of this operation.

(1) If $S \leqq T$, then $T^* \leqq S^*$.

(2) If S is the (set theoretical) join of the subgroups S_α, then S^* is the crosscut of the subgroups S_α^*.

These two properties are immediately obvious. It should be noted, however, that neither the dual nor the converse of (2) is true.

(3) If S is a completely characteristic subgroup of G, then S^* is a strictly characteristic subgroup of G.

PROOF. If $G=G^f$ is satisfied by the endomorphism f of G, if s is an element in S, and if t is an element in S^*, then there exists an element s' in S such that $s=s'^f$; and we find:

$$st^f = s'^f t^f = (s't)^f = (ts')^f = t^f s'^f = t^f s,$$

proving that t^f belongs to S^* too, or $S^{*f} \leqq S^*$.

REMARK. The impossibility of proving that S^* is completely characteristic, if S is completely characteristic, may be shown by the following example: If $S=G$, then S is certainly completely characteristic. But S^* is the center of G; and we have pointed out before that the center is not always completely characteristic.

Note that in the proof of (3) we actually verified the following fact. If S is a subgroup of G, if f is an endomorphism of G, and if $S \leqq S^f$, then $S^{*f} \leqq S^*$. This implies in particular:

(4) S^* is a normal subgroup of G, if S is a normal subgroup of G.

To enunciate the next statement it will be convenient to introduce some notations. If X and Y are subsets of G, then we denote by (X, Y) the subgroup generated by all the commutators $x^{-1}y^{-1}xy$ for x in X and y in Y. For these subgroups P. Hall proved the following important inequality:

(H)[16] If A, B, C are normal subgroups of G, then

$$(A, (B, C)) \leqq (B, (C, A))(C, (A, B)).$$

Now we are able to prove

(5) If $(G, T) \leqq S$ is satisfied by the subgroups S and T of G, then $(S^*, S^*) \leqq T^*$.

PROOF. We deduce from (H) that

$$(T, (S^*, S^*)) \leqq (S^*, (S^*, T)) \leqq (S^*, (G, T)) \leqq (S^*, S) = 1, \text{ proving (5).}$$

We define *the derived series* $G^{(v)}$ of the group G inductively as follows.

(i) $G^{(0)}=G$.

(ii) $G^{(v+1)}$ is the commutator subgroup of $G^{(v)}$.

(iii) If v is a limit ordinal, then $G^{(v)}$ is the crosscut of the subgroups $G^{(u)}$ for $u<v$.

If in particular there exists an ordinal s such that $G=G^{(s)}$, then G may be termed *soluble*.[17]

[16] Hall [1].

[17] A different concept of solubility which is narrower than the present one has been discussed in Baer [1].

THEOREM 5. *Nilpotent groups are soluble.*

PROOF. Put $C_\nu = Z_\nu(G)$* for every ν. Then we have $C_0 = C_1 = G$, since $Z_1(G)$ is the center of G. From $(G, Z_{\nu+1}(G)) \leqq Z_\nu(G)$ and property (5) we deduce that

$$(C_\nu, C_\nu) \leqq C_{\nu+1}.$$

If ν is a limit ordinal, then $Z_\nu(G)$ is the join of the $Z_u(G)$ for $u < \nu$, and we infer from (2) that C_ν is the crosscut of the subgroups C_u for $u < \nu$. Hence it follows by complete (transfinite) induction that

$$G^{(\nu)} \leqq C_\nu \text{ for every } \nu.$$

Since G is nilpotent, there exists an ordinal k such that $G = Z_k(G)$. Clearly C_k is the center of G. But $G^{(k)}$ is part of C_k and C_k is abelian. Hence $G^{(k+1)} = 1$, proving the solubility of G.

Appended to Theorem 1 is a discussion of an example of a nilpotent group G which can be shown to have the property ${}^1G = {}^2G \neq 1$. Thus nilpotency does not imply that the descending central series ends with 1.

On the other hand it may be worth noting that the nonabelian free groups F satisfy ${}^\omega F = 1$, though their center is equal to 1, showing the impossibility of deducing nilpotency from the fact that the descending central chain ends with 1.

We note finally that the situation is altogether different if one is mainly interested in finite chains. For the following theorem is well known:[18] $G = Z_n(G)$ for finite n if, and only if, $1 = {}^m G$ for finite m.

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A SUBSTITUTE FOR THE AXIOM OF CHOICE

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The following result appeared in the 1914 edition of Hausdorff's *Mengenlehre*, p. 140:

(A) *Any partially ordered system contains a maximal simply ordered subsystem.*

This theorem is well known to be equivalent to the axiom of choice (though there does not seem to be a proof of this fact in the literature) and it has been suggested as an alternative for this axiom. The purpose of this note (which is purely methodological) is to propose a simpler but equivalent formulation of (A) as a substitute for the Zermelo axiom. The simplicity lies in the fact that we make no assumptions concerning the relation R which replaces partial order.

Let Q be a set and R an *arbitrary* binary relation on Q. A subset of Q will be termed *R-simple* if for any pair of its elements, a and b, we have either aRb or bRa. The version of (A) we propose is:

(B) *Any R-simple subset of Q is contained in a maximal R-simple subset of Q.*

It is clear that (B) implies (A). Conversely, let Q_0 be an R-simple subset of Q. Let P be the partially ordered (by inclusion) system composed of all R-simple subsets of Q which contain Q_0. Then by (A) there is a maximal simply ordered subsystem P_0 of P. The union of all the sets in P_0 is the desired maximal R-simple subset.

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PROOF OF A THEOREM OF THORIN

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Let

$$A(x, y) = \sum_{j,k=1}^{m,n} a_{jk}x_j y_k$$

be a complex bilinear form, and let $M(\alpha, \beta)$ denote the upper bound of $|A(x, y)|$ for the x's and y's satisfying the condition

$$(\sum |x_j|^{1/\alpha})^{\alpha} \leqq 1, \qquad (\sum |y_k|^{1/\beta})^{\beta} \leqq 1.$$

Then $\log M(\alpha, \beta)$ is a convex function of the point (α, β) in the triangle

$$0 \leqq \alpha \leqq 1, \qquad 0 \leqq \beta \leqq 1, \qquad \alpha + \beta \geqq 1. \tag{1}$$

The above result is due to M. Riesz.[1] In a recent paper it was shown by Thorin[2] that the theorem is restricted neither to bilinear forms nor to the triangle (1). Thorin's result is as follows.

THEOREM. (i) *Let $f(z_1, z_2, \cdots, z_r)$ be an entire function of r complex variables $z_1, z_2, \cdots, z_r$. Let K be a bounded domain $(v_1, v_2, \cdots, v_r)$ of the r-dimensional Euclidean space, satisfying the conditions $v_1 \geqq 0$, $v_2 \geqq 0, \cdots, v_r \geqq 0$. Let $M(\alpha_1, \alpha_2, \cdots, \alpha_r)$ denote the upper bound of $|f(z_1, z_2, \cdots, z_r)|$ for*

$$|z_1| = v_1^{\alpha_1}, \cdots, |z_r| = v_r^{\alpha_r}, \quad \text{and} \quad (v_1, v_2, \cdots, v_r) \in K.$$

Then $\log M(\alpha_1, \alpha_2, \cdots, \alpha_r)$ is a convex function of the point $(\alpha_1, \alpha_2, \cdots, \alpha_r)$ in the domain $0 \leqq \alpha_j < +\infty$, $j = 1, 2, \cdots, r$.

(ii) *If the points of K satisfy a condition*

$$0 < A \leqq v_j \leqq B < +\infty \qquad (j = 1, 2, \cdots, r)$$

then $\log M(\alpha_1, \alpha_2, \cdots, \alpha_r)$ is convex in the whole space $-\infty < \alpha_j < +\infty$, $j = 1, 2, \cdots, r$.

In case (ii), vanishing of some of the α's does not require additional discussion. The situation is slightly different in case (i), since $v_j^{\alpha_j}$ has no meaning if both v_j and α_j are zero. The sets $z_1, z_2, \cdots, z_r$

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[1] M. Riesz, *Sur les maxima des formes bilinéaires et sur les fonctionnelles linéaires*, Acta Math. vol. 49 (1926) pp. 465–497.

[2] G. O. Thorin, *An extension of a convexity theorem due to M. Riesz*, Kungl. Fysiografiska Sällskapets i Lund Förhandlinger vol. 8 (1939) no. 14.

for which this occurs have simply to be disregarded in the definition of $M(\alpha_1, \alpha_2, \cdots, \alpha_r)$.

Owing to the importance of the theorem (even in the special case of M. Riesz), any simplification of its proof may be of interest. The proof given below seems to be slightly simpler than the original one of Thorin, and is based on the following elementary principle.[3]

(a) *A necessary and sufficient condition that a real valued function $\phi(t)$ be convex in an interval (a, b) is that, for any subinterval (a', b') and any real number μ, the maximum of the function $\phi(t)+\mu t$ in (a', b') be attained at least at one of the end points a', b'.*

Its proof is immediate. If $\phi(t)$ is convex, so is $\phi_1(t)=\phi(t)+\mu t$. Since no point of the arc $y=\phi_1(t)$, $a' \leqq t \leqq b'$, lies above the chord joining its end points, the maximum of $\phi_1(t)$ in (a', b') is either at $t=a'$ or $t=b'$. Conversely, if $\phi(t)$ is not convex in (a, b), there is an arc $y=\phi(t)$, $a' \leqq t \leqq b'$, which lies partly above the corresponding chord. If $y=-\mu t-\nu$ is the equation of the chord, the sum $\phi(t)+\mu t+\nu$ vanishes for $t=t_1, t_2$, and takes on positive values at some points inside (a', b'). Hence $\phi(t)+\mu t$ does not attain its maximum in (a', b') at either of the end points a', b'.

The following form of the principle will be more convenient for our purposes.

(b) *A necessary and sufficient condition that the logarithm of a non-negative function $\phi(t)$ be convex in an interval (a, b) is that, for any subinterval (a', b') and any real μ, the maximum of the function $\phi(t)e^{\mu t}$ in (a', b') be attained at least at one of the end points a', b'.*

The necessity of the condition follows from (a). So does the sufficiency, if $\phi(t)$ is strictly positive. It is, however, easy to show that if $\phi(t)$ is non-negative, satisfies the condition of (b), and vanishes at a point t_0 of (a, b), then $\phi(t)$ vanishes identically inside (a, b). For if $\phi(t)$ were positive at a point t_1 which lies, for example, to the right of t_0, so that $t_0<t_1<b$, and if $t_1<t_2<b$, then $\phi(t)e^{\mu(t-t_1)}$ is 0 at t_0, is $\phi(t_1)$ at t_1, and is very small at t_2, if μ is negative and large enough. Hence the maximum of $\phi(t)e^{\mu(t-t_1)}$ in (t_0, t_2) is attained at neither end point. This proves (b), if we agree to consider the function which is $-\infty$ everywhere inside (a, b) as convex.

[3] This principle, both in forms (a) and (b), was first stated by S. Saks and used by him to simplify certain proofs from the theory of functions. See his papers *Sur un théorème de M. Montel*, C. R. Acad. Sci. Paris vol. 187 (1928) pp. 276–277, and *On the properties of convex and subharmonic functions* (in Polish), Mathesis Polska vol. 6 (1930).

We now pass to the proof of the theorem of Thorin, starting with part (ii). Let

$$v_j = e^{\xi_j}, \qquad j = 1, 2, \cdots, r.$$

If $(v_1, v_2, \cdots, v_r) \in K$, the ξ's are bounded. Obviously, we may assume here that K is closed. Hence

$$M(\alpha_1, \alpha_2, \cdots, \alpha_r) = \max \left| f(e^{\alpha_1\xi_1+i\eta_1}, \cdots, e^{\alpha_r\xi_r+i\eta_r}) \right|$$

for $(e^{\xi_1}, e^{\xi_2}, \cdots, e^{\xi_r}) \in K$ and $\eta_1, \eta_2, \cdots, \eta_r$ arbitrary. Since $\left| f(e^{\alpha_1\xi_1+i\eta_1}, \cdots) \right|$ is a continuous function of the variables α_j, ξ_j, η_j, it follows that $M(\alpha_1, \alpha_2, \cdots, \alpha_r)$ is a continuous function of the α's. In order to show that its logarithm is convex on any straight line

$$\alpha_1 = \overset{0}{\alpha_1} + \lambda_1 t, \ \alpha_2 = \overset{0}{\alpha_2} + \lambda_2 t, \cdots, \alpha_r = \overset{0}{\alpha_r} + \lambda_r t,$$

it is enough to show that, for any real μ, the product

$$M(\overset{0}{\alpha_1} + \lambda_1 t, \cdots, \overset{0}{\alpha_r} + \lambda_r t)e^{\mu t}$$

has no proper maximum for any finite t.

Let us assume, contrary to this, that for some μ such a maximum does exist. By a change of variable we may adjust it so that the maximum is attained for $t=0$. Then

$$(2) \qquad M(\overset{0}{\alpha_1}, \overset{0}{\alpha_2}, \cdots, \overset{0}{\alpha_r}) \geqq M(\overset{0}{\alpha_1} + \lambda_1 t, \overset{0}{\alpha_2} + \lambda_2 t, \cdots, \overset{0}{\alpha_r} + \lambda_r t)e^{\mu t}$$

for small $|t|$, and we have strict inequality for some t's arbitrarily small. We may write

$$M(\overset{0}{\alpha_1}, \cdots, \overset{0}{\alpha_r}) = \left| f(e^{\overset{0}{\alpha_1}\overset{0}{\xi_1}+i\overset{0}{\eta_1}}, \cdots, e^{\overset{0}{\alpha_r}\overset{0}{\xi_r}+i\overset{0}{\eta_r}}) \right|,$$

where $(e^{\overset{0}{\xi_1}}, e^{\overset{0}{\xi_2}}, \cdots, e^{\overset{0}{\xi_r}})$ is a point of K, and the $\eta_1^0, \eta_2^0, \cdots, \eta_r^0$ are suitably chosen. Thus

$$(3) \qquad \left| f(e^{\overset{0}{\alpha_1}\overset{0}{\xi_1}+i\overset{0}{\eta_1}}, \cdots, e^{\overset{0}{\alpha_r}\overset{0}{\xi_r}+i\overset{0}{\eta_r}}) \right| \geqq \left| f(e^{(\overset{0}{\alpha_1}+\lambda_1 t)\overset{0}{\xi_1}+i\eta_1}, \cdots, e^{(\overset{0}{\alpha_r}+\lambda_r t)\overset{0}{\xi_r}+i\eta_r})e^{\mu t} \right|$$

for all real t's sufficiently small in absolute value, and for all real η's. We assert that (3) holds for all *complex* t's of sufficiently small modulus. This follows immediately, for if we replace t by $t+iv$ in (3), the only effect will be to change the η's. It follows in particular that the modulus of the entire function (in t)

$$f(e^{(\overset{0}{\alpha_1}+\lambda_1 t)\overset{0}{\xi_1}+i\overset{0}{\eta_1}}, \cdots, e^{(\overset{0}{\alpha_r}+\lambda_r t)\overset{0}{\xi_r}+i\overset{0}{\eta_r}})e^{\mu t}$$

attains a proper maximum at $t=0$, which contradicts the principle

of the maximum for analytic functions. Hence part (ii) of the theorem is proved.

As Thorin himself points out, part (i) of the theorem is an immediate consequence of part (ii). For if K_ϵ is the part of K for which $v_j \geqq \epsilon$, $j=1, 2, \cdots, r$, and if $M_\epsilon(\alpha_1, \alpha_2, \cdots, \alpha_r)$ is the function $M(\alpha_1, \alpha_2, \cdots, \alpha_r)$ corresponding to K_ϵ, then

$$\lim_{\epsilon \to 0} M_\epsilon(\alpha_1, \alpha_2, \cdots, \alpha_r) = M(\alpha_1, \alpha_2, \cdots, \alpha_r).$$

This holds both when all the α's are different from zero and when some of them vanish. Hence $M(\alpha_1, \alpha_2, \cdots, \alpha_r)$ is convex for $\alpha_1 \geqq 0, \alpha_2 \geqq 0, \cdots, \alpha_r \geqq 0$. The restriction $\alpha_j \geqq 0$ is introduced to avoid infinite values for M.

Let $k>0$. The theorem of Thorin may be considered as the limiting case $(k=+\infty)$ of a similar theorem in which the expression $M(\alpha_1, \alpha_2, \cdots, \alpha_r)$ is replaced by

$$M_k(\alpha_1, \alpha_2, \cdots, \alpha_r) = \max \int_0^{2\pi} \cdots \int_0^{2\pi} \left| f(e^{\alpha_1 \xi_1 + i\eta_1}, \cdots, e^{\alpha_r \xi_r + i\eta_r}) \right|^k d\eta_1 \cdots d\eta_r$$

and $(e^{\xi_1}, \cdots, e^{\xi_r}) \in K$. The limitations on the α's are the same as before. The proof does not differ from the one given above, if we use the fact that the integral

$$\int_0^{2\pi} \cdots \int_0^{2\pi} \left| f(e^{(\alpha_1^0 + \lambda_1 t)\xi_1^0 + i\eta_1}, \cdots, e^{(\alpha_r^0 + \lambda_r t)\xi_r^0 + i\eta_r} \right|^k d\eta_1 \cdots d\eta_r$$

is a subharmonic function of the (complex) variable t, and so cannot attain a proper maximum. That this integral is a subharmonic function of t follows from the fact that $\left| f(e^{(\alpha_1^0 + \lambda_1 t)\xi_1^0 + i\eta_1}, \cdots) \right|^k$ is subharmonic in t for every fixed set of $\eta_1, \eta_2, \cdots, \eta_r$.

As Thorin pointed out, the function $f(z_1, z_2, \cdots, z_r)$ of his theorem need not be entire. If it is regular in a domain D, the function $M(\alpha_1, \cdots, \alpha_r)$ is convex in a domain of the variables $\alpha_1, \cdots, \alpha_r$. The same remark applies to M_k.

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